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13. ABSTRACT

Many experiments involve measuring a number of response variables simultaneously. As a result of giving one stimulus to an experimental unit, what we obtain is not just one response but several responses. In statistical language, we deal with a multivariate situation as opposed to univariate situations. Usually, many stimuli, called factors, are considered at many levels in the same experiment. Many statistical techniques are available to analyse this type of data and to draw conclusions therefrom. The present work considers one such technique, the identification of subgroups of individuals on the basis of responses, i.e., a special case of cluster analysis. Many different algorithms proposed for detecting clusters have been reviewed. These fall into two classes--(i) those which detect clusters of variables--factor analysis--and (ii) those which detect clusters of experimental units--cluster analysis. What we have done in the present work lies in-between these two techniques. We first perform cluster analysis on p responses for each unit, and sort the experimental units into groups. After assigning experimental units to groups, we look for those individuals whose total response could be expressed in (p-1) combinations of original responses, irrespective of the groups or clusters to which they belong. Those individuals whose total response could be described in terms of (p-1) combinations of original responses are said to lie on a "simple structure plane." A geometric probability approach has been used to determine whether a given point lies on a simple structure plane. The orthogonal distance of a point from the plane is used as a criterion. If many points lie on a given simple structure plane, the plane is said to be overdetermined. Probability expression has been derived to determine whether a simple structure plane can be regarded as being overdetermined. Those individuals which lie on a (p-1)-dimensional hyperplane in a p-dimensional space, need one variable less in their description.

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13. ABSTRACT

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In this case, (p-1) linear combinations of the original p variables, could be used for describing these points. For real life interpretation, this procedure would be useless as we would be left with (p-1) artificial variables instead of p observable variables. It would seem reasonable, then, to eliminate, for the data points close to one of the subspaces, that observable variable which contributes the least to the description of this selection of data points. A correlation approach has been used to determine the variablé to be eliminated, and it turns out that the variable which correlates most strongly in absolute value with the artificial variable should be discarded.

Two computer programs have been developed to assist the user in analysing his data using this approach. The first one, named CLUSTR, identifies clusters and simple structure planes. The second one, an interactive graphics program, named ELLIPSE, can be used to visualize the configuration of clusters and the underlying simple structures. The clusters are projected onto many 2-dimensional spaces and displayed on the IBM 2250 Graphics terminal. If the plane of projection selected is orthogonal to a simple structure plane, the points lying on the particular simple structure plane, will make a band of narrow width more or less resembling a straight line. For other planes of projection this will not hold. The clustering of points is, however, unaffected by the choise of a particular plane of projection.

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# CONFIGURATION AND CLASSIFICATION OF CLUSTERS IN n-DIMENSIONS

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# CHAPTER I

# INTRODUCTION

## 1.1 Multivariate Experiments

Many experiments involve measuring a number of response variables simultaneously. Thus, for example, in determining the effectiveness of a new drug, a person's systolic and diastolic blood pressure may be observed, before and after administration of the drug; also his pulse rate, temperature and other physiological data may be recorded. As a result of giving one stimulus to an experimental unit, what we obtain is not just one response but several responses. In statistical language, we deal with a multivariate situation (many responses) as opposed to univariate situations (only one response). An experiment is rarely so simple as described above. Usually, many stimuli, which will be called factors, are considered at many levels in the same experiment. Ultimately, the experimenter has a large collection of data before him. The problem is how to interpret the data. Depending upon the experimenter's objective, many statistical techniques are available to analyse the data and to draw conclusions therefrom. In the present work, we consider one such technique, the identification of subgroups of individuals on the basis of responses, i.e., a special case of cluster analysis. In the following sections, we review some of the problems of the subject and then outline the special problem in the present dissertation.

## 1.2 Review of the related literature

Consider an experiment where many responses are recorded on each experimental unit. We shall assume that there are n experimental units, and, on each unit, p responses are measured. The resulting data can be put in the form of an $n \times p$ matrix. Each row of this matrix corresponds to one experimental unit. Each experimental unit can be represented by a point in a p-dimensional space. It is possible that some of these points will be so close to one another that they form a "cluster." The problem of detecting clusters has been considered by many authors during the last 30 years. Tryon [37] in 1939 gave many algorithms based on the correlation matrix of variables, for the related problem of assigning variables to groups. The technique, much similar to the concept of the "coefficient of belonging" described by Harman [14], was developed on the assumption that correlations among variables belonging to the same group should be much higher than correlations between these variables and those not belonging to the group. Holzinger and Harman [15] defined their coefficient of belonging or B-coefficient as "100 times the ratio of the average of the intercorrelations among the variables of a group to their average correlation with all the remaining variables.

Sokal and Michener proposed the weighted mean-pair method to identify clusters. This method was originally applied to an entomological problem [33]. A more recent description of this method has been given by Sokal and Sneath [34] who recommend it as "the best of a class of commonly used methods of cluster analysis." The method operates on an $N \times N$ similarity matrix. Those two individuals, i and j say, which have the highest similarity are paired together, i.e., put in the same cluster.

Any appropriate measure of similarity e.g., product moment correlation coefficient, coefficient of association, etc., can be used, although by far the most commonly used measure is the product moment correlation coefficient. After the individuals i and j have been paired together, columns (and rows) i and j of the similarity matrix are replaced by a single column consisting of the means of the elements in rows (and columns) i and j. The process is then repeated on the new matrix of order (N-1), when either two new individuals have the highest similarity and form a new pair, or the existing pair combines with a further individual to make a cluster of three. The process continues and at each stage a new cluster consisting of a pair of individuals is formed or the new individual is assigned to one of the already existing clusters of previously combined individuals.

Edwards and Cavalli-Sforza [7] suggest dividing the points into two sets such that the sum of squares of distances between sets is a maximum. Thus according to this method, one can find only two clusters, no more and no less. Since the total sum of squares is a constant for a given sample, maximizing between-groups sum of squares is equivalent to minimizing the within-groups sum of squares. The method consists of examining all the $2^{N-1} - 1$ two-set partitions of N individuals and selecting the one which gives the minimum within-set sum of squares. The method is not suitable for a large value of N as the time required on a computer to examine all the two-set partitions is enormous. It was estimated that with N = 21, the time required to examine all the partitions, on a computer with 5 micro-second access time would be 100 hours and that for N = 41, it would be 54,000 years. Thus even with the help of the fastest computers, the method would be impracticable. One more algorithm based on ecological applications was proposed by Williams and Lambert [38]. Gower [12] gives an excellent comparison of the last three mentioned algorithms together with some of his own modifications proposed for these algorithms.

Another important study was made by Neyman and Scott [27]. They extensively studied the clustering of galaxies in the universe and presented the theories of "simple clustering" and "multiple clustering." "Simple clustering" was based on the assumption that galaxies occur in clusters and that the cluster centers are uniformly distributed throughout the universe. In "multiple clustering" it was assumed that the cluster centers radiate from super clusters.

From the above discussion, it is clear that in the statistical literature, we come across two types of clusters--(i) the clusters of variables and (ii) the clusters of individuals or points or experimental units. Without going into the details of the confusion that these two concepts have created and their uses and misuses, we only note that given a multivariate sample, it is possible to identify the underlying clusters by applying any of the suitable techniques available.

We now describe in detail an algorithm proposed by Bargmann and Graney [5] to determine clusters with the object of identifying mixed samples of multivariate normal distributions. With the help of methods to be developed in the present work, we may then study the configuration of such subgroups.

## 1.3 Real and Virtual Clusters

Real clusters are defined to be clusters of points in the original space. Virtual clusters on the other hand are clusters of projections of points in a space of dimension lower than that of the original space. To borrow an example from Graney [10], if one were to look for clusters of stars as observed from the earth, one would be dealing with virtual clustering. The observer perceives the stars as projections onto the surface of the celestial sphere. To determine the real clusters of stars, it would be necessary to measure the distance of each star from the observer. It is also obvious from the above example that, virtual clusters may not necessarily be real clusters, and vice versa. One may tend to think that real clusters would necessarily be virtual clusters also. This, however, is not so. If one were standing in the midst of a real cluster, he may not find any cluster at all. In this connection, it should also be noted that in the algorithm proposed by Graney [10], if a cluster is centered at the center of the entire system, it would not be detected. This is similar to identification of galaxies. Being a member of our own galaxy, we do not obtain, by direct observation, a description of the configuration of our galaxy. For that purpose, we will have to make calculations based upon distance measurements (or observe from a different galaxy).

## 1.4 d-clusters and k-clusters

As experiemental units are assigned to clusters, a decision has to be made whether two units are close enough to justify their inclusion in the same cluster. One may look at this problem in two directions: The so-called d-clusters and k-clusters. Consider a region S of fixed radius d. This is said to be overdetermined at the α-level of significance if the number of points in the region exceeds a value $k_o$ such that, under the null hypothesis of uniform distribution of points,

$$P\,[k_o \text{ or more points in } S] < \alpha$$

This type of cluster is known as d-cluster since it results from a region of fixed radius d.

The other type of clustering results when a fixed number of points fall into a region with sufficiently small radius. Let there be a fixed number of points, say, k. Let $d_o$ be the radius of a sphere (or hypersphere) just sufficient to enclose all these k points within the sphere. It is apparent that d, the radius of the sphere is a random variable. If

$$P\,[d < d_o] < \alpha$$

where $\alpha$ is the predetermined level of significance, then these k points are said to form an overdetermined cluster. Such a cluster is known as the k-cluster since it results from a fixed number of points falling within a sufficiently small sphere.

## 1.5 Identification of Mixed Samples

Consider the case of k-cluster discussed above. As an example of this type of cluster, consider an experiment in which the number of people given a drug from a certain class of drugs (which produce similar effects) is limited. Then one would like to know if the symptoms among some individuals are more closely alike than those of other individuals. In other words, it would be appropriate to see if there are some individuals whose symptoms are so close that they form a cluster. But this also implies that we are looking for a principle of classification which distinguishes this group of individuals. Such a situation can arise in linear analysis. For the sake of simplicity, we shall describe the situation in terms of univariate analysis, but it applies equally well to multivariate analysis. In a two-factor experiment one being applied at r levels and the other being applied at s levels, we will have r x s cells; let us assume that there are $n_{ij}$ observations in cell (i, j). Analysis of these data on the basis of a linear statistical model assumes homogeneity of cell variances. The hypothesis of the equality of cell variances can be tested by application of Bartlett's test. If this hypothesis should be rejected, three possible causes can be responsible: (i) The cell variances are not constant, $\sigma^2$, but proportional to some known $v_{ij}$ (a different one for every cell). A variance-stabilization transformation, or weighted regression, or both, can be employed to correct this situation. (ii) There may be "mavericks" --misclassified or improperly recorded items. They can be omitted before the analysis of the data. (iii) There may be a third factor of classification present. If this is the case, what we regard as "error sum of squares" is in fact not the error sum of squares but the variance component sum of squares error, plus sum of squares due to the third factor which we have not taken into account. In multivariate analysis of variance, this quantity would be the (H + E) matrix instead of the E matrix alone where E stands for the "error" SSP matrix and H stands for the "hypothesis" SSP matrix.[1] Hence the problem reduces to that of detecting the third hidden factor. It is clear that the sample within each cell comes from two or more populations instead of from just one. It will be necessary to "unmix" the samples within each cell before a valid linear statistical analysis of the data can be performed. Instead of describing the technique of unmixing the mixed samples, we refer to Graney [10], for a full description of the technique, which also contains a number of illustrations.

[1]SSP sum of squares and products, sometimes also called "Wishart" matrix.

There has been, in recent years, a considerable resurgence of interest in cluster analysis. Ling [23] has discussed many techniques and he has tried to classify these. It is apparent that there is little purpose in inventing yet new similarity indices, distance measures, or search algorithms. The present dissertation deals with a point intermediate to the two problems which cluster analysis has attempted: (a) classification of individuals, on the basis of responses and (b) classification of response variables assuming a homogeneous group of individuals (really factor analysis). Cattell [6], and Stephenson [35] view the entire complex as "factor analysis" and call the first problem the "Q technique," the second problem the "R technique," and the combined problem, the "P technique." Unfortunately, their techniques do not lend themselves to a study of configurations because in their attempt to regard every problem as a correlational one, the authors perform analyses which become self-contradictory. For example, sums of squares and products can be used as terms in estimates of correlation between variables only if the individuals are <u>independent</u>, and conversely, "correlations" between individuals can be estimated by a product-moment approach only if the variables are uncorrelated. Quadratic forms would be needed otherwise, and the sums of squares and products can be quite meaningless. The present study avoids this confusion by separating, at each stage, the clustering problem (cluster of individuals) from the problem of structures of <u>variables</u> within clusters.

Guttman [11] has proposed a yet another technique to reduce the dimensionality of data. The technique, "nonmetric" in nature, works on an n x n symmetric matrix R, and determines those transformations which yield an Euclidean coordinate system X (X : n x m), such that XX' = F for m a minimum and, furthermore, satisfy all inequalities that whenever $r_{ij} > r_{kl}$ then $f_{ij} > f_{kl}$ for the non-diagonal elements of R and F, ($i \neq j$, $k \neq l$). This avoids the problem of communalities and "when some lawful structure or pattern is present in the data, e.g., a simplex, a circumplex, or a radex, a nonmetric analysis will reveal the configuration, whereas a metric approach will obscure the lawfulness." For detailed discussion of this approach and the algorithms developed in this connection, refer to Guttman [11] and Lingoes and Guttman [24]. It is clear that this technique will reduce the dimensionality of all the data points. Again, as in factor analysis, it will not be affected by mean shifts. It is thus an alternative to structural and factor analyses and not an "in-between" solution as proposed by us in section 1.6.

## 1.6 Definition of the Problem:

In the above sections, we have given a brief outline of traditional approaches to cluster analysis. The discussion reveals that, given a multivariate sample, it is always possible to detect some underlying cluster structure and to assign points (experimental units) to the clusters to which they belong. This is not the only kind of analysis that can be performed. The same data could also be subjected to factor analysis, which assigns response variables to classes. Cluster analysis will group the individuals bringing out the mean effects and leaving the variable structure unaltered. Factor analysis will describe the data in terms of artificial variables without telling anything about the group means involved. What we propose to do in the present work lies in-between these two extremes. We first perform cluster analysis on p responses for each unit, and sort the experimental units into groups. After assigning experimental units into groups, we look for those individuals whose total response could be expressed in (p-1) combinations of original responses, irrespective of the groups or clusters to which they belong. Thus the ultimate purpose of our analysis is to elicit more information from a set of multivariate data than is possible by cluster or factor analysis alone. Frequently both of these extremes produce trivial results. In medical applications, cluster analysis tends to producing clusters of patients who are healthy, slightly ill, and very ill. By contrast, factor analysis identifies a collective set of symptoms such as "fever," "pain," and "chills."

The algorithm developed in the present work begins by identifying clusters in the sense of estimating parameters of mixed multivariate normal distributions with equal variance-covariance matrices. If this were not done the unit ellipsoids around the grand mean would be affected, uncontrollably, by mean shifts. Within each of these ellipsoids, we look for well overdetermined subspaces of lower dimensionality, in the sense that we want a very significant number of individuals to fall into a region close to these highly overdetermined subspaces. These will be called "Simple Structure Planes."

We use this term because of its similarity with one criterion which Thurstone [36] used in describing a "simple structure" in the common-factor space of factor analysis. If all principal components of the within-cluster matrix were calculated and "rotated," the results of our study could also be produced. This, however, would be a tremendously wasteful procedure.

Those individuals which have a large distance from the subspace, yet belong to the original cluster, would differ from the others in that they require at least one additional diagnostic.

Geometrically, the Euclidean distance on a metric given by the unit ellipsoid, is obtained as follows: A vector is passed from the center of the ellipsoid (O) to the point in question (A). This vector intercepts the unit ellipsoid at point (S). The Euclidean distance is then (length OA)/(length OS). For this reason, the ellipsoid is called "unit ellipsoid." It generalizes the concept of a "unit interval" in one dimension (hence metric). The computer output reports these distances as a vector $\underline{v}$.

Now, the largest projected distance of a point, onto a 2-dimensional subspace, from the simple structure unit ellipsoid, is at most equal to the actual distance in p dimensions. Hence a point showing an appreciable distance from the simple structure ellipsoid, on any projection, would be representative of a point requiring one variable more, for adequate description, than the points lying in the simple structure region. This property is not related to the clustering of the points. Points in the same simple structure subspace may be far removed from each other; they may belong to different clusters.

The points having a large (positive or negative) distance from each of these planes can now be scrutinized. They share some characteristics, which makes them different from the other points on this plane. It is to be noted here that this procedure was not intended to find subclusters--one could do that by tightening the control constants in the original program--but to study subspaces of lower dimensionality.[2] Thus the multivariate data are viewed from a new point of reference, and one may identify principles permitting a different taxonomy of experimental units (patients, plants, etc.).

There is a certain analogy of techniques between the subspace solution and the "simple structure rotation" in factor analysis. This is expected in virtual clusters. The cosine of the angular distance between two vectors can be regarded as a correlation if the vectors represent variables. It is in this sense that our n data points correspond to n correlated variables of factor analysis and our variables themselves correspond to the factors. With this understanding we can apply the rotational techniques to the original data matrix in order to obtain points lying on simple structure planes.

In the present dissertation, we have synthesized these two techniques--cluster analysis and simple structure identification--into a single program. When the user subjects his data to this program, named CLUSTR, he gets clusters and simple structure planes as the output. Next, we have developed an interactive graphics program, named ELLIPSE, which can be used to visualize the configuration of clusters and the underlying simple structures. Configuration of multidimensional clusters can be determined only if their dimensionality is reduced. For this purpose, it is necessary to project the clusters onto several 2 or 3 dimensional subspaces. The emphasis in the present work is on projecting the clusters

[2]Those readers who assume that this is analogous to factor analysis should be reminded that the latter increases the dimensionality from p correlated to p + k (at least partially) uncorrelated variables. There is, of course, no relationship to an incomplete "component analysis" which produces singular solutions.

onto many 2-dimensional spaces and displaying them on the IBM 2250 Graphics terminal. If the plane of projection selected is orthogonal to a simple structure plane, the points lying on the particular simple structure plane will make a band of narrow width more or less resembling a straight line. The display program can also be used in an exploratory manner. The user can supply many different vectors. If, by using some vector of projection, he sees narrow bands as described above, the corresponding vector is orthogonal to a simple structure plane. Such visual determination of simple structure planes, however, is rather difficult especially if the user is required to make inferences on configurations in four or more dimensions, on the basis of 2-dimensional displays. It is implicit from the above discussion that the determination of simple structure is equivalent to the fact that the points lying on a simple structure plane need at least one variable less in their description. Thus if p variables have been measured on all the experimental units of the sample (p-1) variables are adequate in the case of those experimental units which lie on a simple structure plane. In the case of these experimental units, one of the p variables can then be expressed by a linear combination of the remaining (p-1) variables. The statistical interpretation of this phenomenon is considered in Chapter 3.

# CHAPTER II

# PRINCIPAL COMPONENT AND OTHER PROJECTIONS

## 2.1 Introduction

As stated in the previous chapter, one of the goals of the present work is to display many different projections of multidimensional clusters. In this chapter, we give an account of projections along eigenvectors or principal components of the metric ellipsoids onto 2-dimensional subspaces. The principal component projections are necessarily "orthogonal." It is worth noting at the outset that very little information was gained by these principal component projections. Not that we were surprised by this finding, but some social scientists seem to attribute a lot more to this particular mathematical reference frame than it deserves.

## 2.2 Principal Component Projections

Let $Z = (Z : n \times p)$ be a data matrix of the multivariate observations. The $n$ data points can be represented in a p-dimensional space. It is assumed that the clusters formed by these $n$ points have been identified as well as the points belonging to the clusters. It is further assumed that points, which cannot be assigned to one of these clusters, have been eliminated. Now, if the data come from a p-variate normal distribution (or a mixture of p-variate normal distributions with different mean vectors but the same variance-covariance matrix), the clusters would be elliptical in shape, and the ellipsoids would have the same orientation. If there are k clusters, and if we denote by $\underline{\mu}_1, \underline{\mu}_2, \ldots, \underline{\mu}_k$ the cluster centers, the equations of the ellipsoids enclosing these clusters can be written as

$$(\underline{x} - \underline{\mu}_i)' \Sigma^{-1} (\underline{x} - \underline{\mu}_i) = \text{constant}$$

for i = 1, 2, . . . , k, where $\underline{x}$ stands for the running coordinates. The points belonging to a particular cluster will be enclosed within the respective ellipsoids. In practice, since the population (or populations) from which the sample was drawn, will not be exactly normal, we will not expect all the data points belonging to a particular cluster to lie within-the corresponding ellipsoid. However, unless the population is far from normal, the ellipsoidal fit will be quite good. To visualize how the points are distributed in p-dimensional space and how they look in relation to their enclosing ellipsoids, we need to project the points onto several 2-dimensional spaces. The technique is simple and classic and is spelled out here for the sake of completeness. Let us take the equation of the ith unit ellipsoid[1]

$$(\underline{x} - \underline{\mu}_i)' \Sigma^{-1} (\underline{x} - \underline{\mu}_i) = 1 \qquad (2.2.1)$$

Since $\Sigma$ is a positive definite ( or at least positive semi-definite) matrix. there exists an orthogonal matrix Q such that

$$Q' \Sigma Q = D_\lambda \qquad (2.2.2)$$

where $D_\lambda$ is a diagonal matrix with diagonal elements equal to the characteristic roots of $\Sigma$, and Q is the matrix of the eigenvectors of $\Sigma$. (2.2.2)

[1]This generalizes the "standard unit interval" in univariate analysis.

can be written as

$$\Sigma = Q\, D_\lambda\, Q' \tag{2.2.3}$$

and hence

$$\Sigma^{-1} = Q\, D_{1/\lambda}\, Q \tag{2.2.4}$$

where $D_{1/\lambda}$ is the inverse of $D_\lambda$. (The reciprocals of the characteristic roots of $\Sigma$ are the diagonal elements of $D_{1/\lambda}$). Substituting (2.2.4) into (2.2.1), we have

$$(\underline{x} - \underline{\mu}_i)'\, Q\, D_{1/\lambda}\, Q'\, (\underline{x} - \underline{\mu}_i) = 1 \tag{2.2.5}$$

or, if we let $\underline{y}'_i = (\underline{x} - \underline{\mu}_i)'\, Q$, this reduces to

$$\underline{y}'_i\, D_{1/\lambda} \underline{y}_i = 1 \tag{2.2.6}$$

(2.2.6) is the standard principal axes reduction of the conic (2.2.1). The transformation $\underline{y}'_i = (\underline{x} - \underline{\mu}_i)'\, Q$ is the orthogonal rotation of the original reference axes in the direction of the principal axes of the ellipsoids. The direction cosines of the principal axes of all the k ellipsoids are identical because of the assumption of equal metric (homogeneity of dispersion matrices). If we write the matrix Q as

$$Q = [\underline{q}_1, \underline{q}_2, \ldots \underline{q}_p]$$

where $\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_p$ are the eigenvectors of $\Sigma$, the transformation $\underline{y}'_i = (\underline{x} - \underline{\mu}_i)'\, Q$ can be written as

$$\underline{y}'_i = (\underline{x} - \underline{\mu}_i)'\, [\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_p].$$

Since Q is orthogonal, $\underline{z}'_i\, Q$ will be coordinates of an original data point $\underline{z}'_i$, which is the ith row of the data matrix Z, with reference to the new coordinate axes. In particular $\underline{z}'_i \underline{q}_1$, $\underline{z}'_i \underline{q}_2$ will represent the orthogonal projection of the original data point $\underline{z}'_i$ onto the 2-dimensional plane determined by the eigenvectors $\underline{q}_1$ and $\underline{q}_2$. Let us assume that the eigenvectors are arranged in descending order, i.e., $\underline{q}_1$ is the eigenvector corresponding to the largest root, $\underline{q}_2$ that corresponding to the second largest root, etc. Then the 2-dimensional plane determined by $\underline{q}_1$ and $\underline{q}_2$ is the plane containing the two largest principal axes of the ellipsoids and we would be projecting the data points onto this plane. We can take all the $\binom{p}{2} = p(p-1)/2$ pairs of eigenvectors and project the data points on these planes.

## 2.3 Principal Component Displays

The IBM 2250 Graphics terminal was used to display projections on the p(p-1)/2 2-dimensional planes formed by each pair of the eigenvectors. The data used by Graney [10] were taken, and the three clusters identified by him were projected on all the three 2-dimensional pairs formed by the 3 variables. The unit ellipses (i.e., projections of the unit ellipsoid) around the clusters were also displayed. The orientation of these ellipses, is, in the case of principal components, of course parallel to the axes of reference. If, however, one of the axes is not a principal component, the inclination of the principal axes of the ellipses with the reference axes can be displayed. This inclination may present some evidence regarding the nature of the data points, which was obscured in the principal component plot. Because of the disappointing lack of information contained in the principal component plots, we did not even bother to document the computer program. Instead, the program which has been documented permits projection around arbitrary pairs of reference axes. These computer programs are described in detail in Chapter V.

## 2.4 Other Projections

Let the equation of the n-dimensional unit ellipsoid be

$$(\underline{x} - \underline{\mu}_i)' \Sigma^{-1}(\underline{x} - \underline{\mu}_i) = 1$$

where $\Sigma$ is the p x p variance-covariance matrix, $\underline{x}$ are the running coordinates and $\underline{\mu}_i$ is the cluster center of the ith cluster. There will be as many ellipsoids as the number of clusters identified. For the sake of notational convenience, we will drop the subscript i from $\underline{\mu}_i$. As all the ellipsoids are referred to the same metric $\Sigma^{-1}$, they differ only with respect to their location. In the discussion that follows, we are concerned with the shape rather than the location. The matrix $\Sigma$, as a population parameter, is unknown and we will replace it by its unbiased estimate, s, the matrix of mean squares and mean products within groups. The projection on the 2-dimensional plane formed by the variables i and j can be written as

$$s^{ii}(x_i - \mu_i)^2 + s^{jj}(x_j - \mu_j)^2 + 2\, s^{ij}(x_i - \mu_i)(x_j - \mu_j) = 1$$

where $s^{ij}$ is the (i, j)th element of $s^{-1}$. These equations for various values of i and j are employed in displaying the projections. For the purposes of computer programming, this equation is further simplified into

$$s^{ii}y_i2+ s^{jj}y_j^2 + 2s^{ij}y_iy_j = 1$$

where $x_i - \mu_i = y_i$ and $x_j - \mu_j = y_j$. By employing the standard reduction techniques, the coordinates to plot the ellipses can be easily calculated. Further aspects on this phase of the computer program are treated in Chapter V.

# CHAPTER III

# CLUSTERS IN SUBSPACES--THE SIMPLE STRUCTURE SUBSPACE

## 3.1 The Problem

The previous two chapters considered the problem of cluster identification and cluster configuration. We now come to the second part of our inquiry--identification of experimental units or points which could be described by measuring a fewer number of variables on them. Before we proceed further with the identification of the points lying on a subspace, we want to consider the situations where such a problem can arise. A familiar example would be one of medical diagnosis. A number of patients are measured on a number of medical symptoms. Sometimes these measurements are repeated on the same patients for a number of days consecutively. Here the symptoms measured are our variables and the patients are subjects or experimental units. If the measurements are taken on different days, it would add a factor of classification; let us assume that this factor (i.e., days) has not been recorded. Of course, from these data one can construct a variance-covariance matrix and from that obtain a correlation matrix. This could be subjected to factor analysis. Factor analysis would reveal groupings of the symptoms in these data. Application of Thurstone's [36] principle of simple structure could reveal such groupings. Disappointing examples of this kind of analysis resulting in the symptoms like "chill," "fever," and "pain" are not so uncommon.

The data could also be subjected to cluster analysis to form groups of patients. In this type of analysis, the patients would be classified into groups depending upon the absence or severity of symptoms they have in common with respect to other patients belonging to the same group. Thus, for the patients belonging to the same group, almost all the patients would be measuring equally on the average on different symptoms; they may either measure high on the same symptom compared to other patients belonging to different groups or measure low, etc. To summarize, factor analysis would tell us about the symptoms, and cluster analysis would help us to group patients according to the "degree of severity," say, of the symptoms. According to cluster analysis, we may have two patients belonging to different groups perhaps because one measured low on one symptom and the other measured high on the same symptom. It may happen that the particular symptom would have left the final diagnosis unchanged. To this extent, this symptom could be discarded so far as these two individuals are concerned and then they will belong to the same "group." But neither the factor analysis nor cluster analysis would bring out this fact; nor would it bring out the fact that "days" is a hidden factor. We must extend both techniques before we can identify such configurations. The problem is formulated in mathematical terms in the next section where we present, in detail, a specific approach.

## 3.2 Mathematical Formulation

In previous chapters we have dealt with the techniques of cluster analysis. It was pointed out that given an $n \times p$ data matrix, it is possible to identify the underlying clusters. We now ask the next question--are there any data points which instead of lying in the p-dimensional space, lie on the (p-1) dimensional hyperplane? This question is important as an answer to it, among other things, will reveal the following things: (i) The points that lie on the (p-1)-dimensional hyperplane. The determination of the points lying on the (p-1)-dimensional hyperplane will help us to describe these points in terms of (p-1) variables instead of p variables. (ii) We will essentially devise a "discriminant function" which helps us to split the original sample into two groups of points--one group which needs all the p original variables to describe the points belonging to it and another group which needs (p-1) instead of p variables to describe the points belonging to it. In the second case, we have also to consider the question of which variable to discard from the original p variables. Note that it is also implied in the second case that the p x p variance-covariance matrix of the points lying on the (p-1)-dimensional hyperplane will be singular, as its rank will be (p-1) and not p.

## 3.3 A Basic Probability

At the outset, let us make clear what we mean by points lying "approximately" in a subspace. Our attempt will be to look for those points which lie close to a (p-1)-dimensional hyperplane in a p-dimensional space. Clearly, any (p-1) vectors always lie exactly on a (p-1)-dimensional hyperplane, whereas a minimum of p vectors is necessary to overdetermine a (p-1)-dimensional hyperplane. We will, therefore, look for those (p-1)-dimensional hyperplanes which have p or more vectors lying close to them. The singular case where p or more vectors lie, exactly, on a plane will be ignored, since in this instance, those points lying on the hyperplane will satisfy a linear relationship between the p-variables; this cannot happen unless there is a deterministic linear relationship between the p-variables in the population, and thus, any sample will reflect this relationship and the $p \times p$ estimated variance-covariance matrix of any sample from such a population will be singular. Since we require inverses of dispersion matrices, we must exclude these redundancies. To determine, whether a point overdetermines a hyperplane, we shall consider its Euclidean distance from the hyperplane and examine whether this distance could be considered negligible in a probabilistic sense. We need an expression for this probability. The derivation follows the reasoning given by Bargmann [2]. Let P be a point in 2-dimensional space. We are interested in the orthogonal distance of this point from a line, which is a hyperplane in 2-dimensional case. Without loss of generality, we can assume the X-axis to be this line (Figure 3.3.1). Let the vector OP subtend an angle $\theta$ at the origin with the X-axis. We must now assume that the reference axes span a Cartesian frame (orthogonal, equal units along each axis). This implies that a Gram-Schmidt transformation of the original observations (using S, the within estimated dispersion matrix as the original metric) must precede the calculation of probabilities of overdetermination. With this assumption we can now draw a circle with OP as radius. Let M be the foot of the perpendicular drawn from P on the X-axis. Then

$$PM = OP \cdot \sin \theta$$

Therefore, $\theta = \sin^{-1} PM/OP = \sin^{-1} 2PM/2OP = \sin^{-1} a/2h$

Where $a = PP'$ and h is the radius of the circle. Thus, the probability that a point falls within the angle $\theta$ on the circumference of the circle is given by

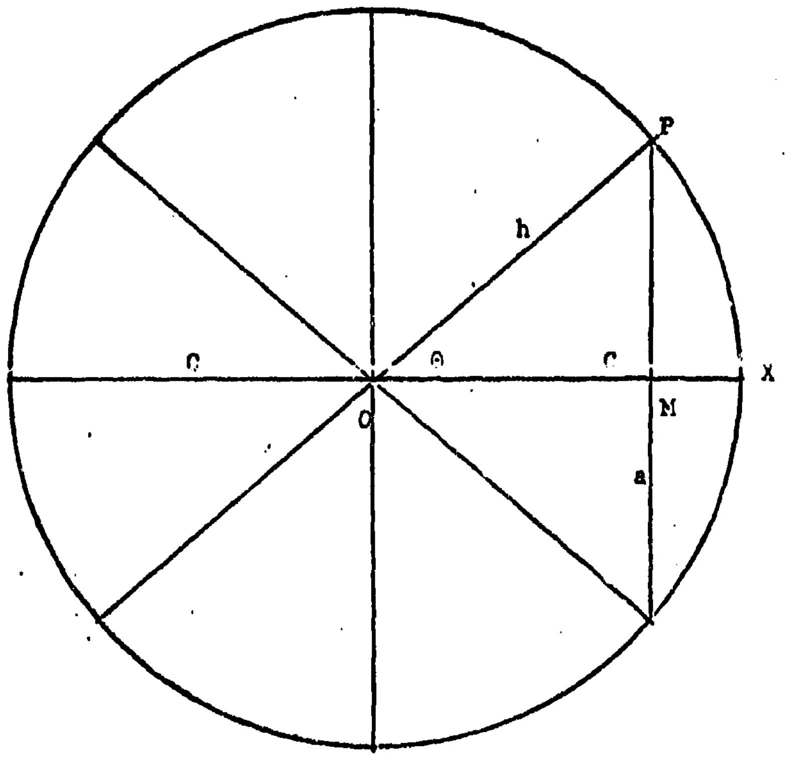


Figure 3.3.1

$$P_2 = \frac{2 \times \text{length of arc generated by angle } 2\Theta}{\text{circumference of the circle}}$$

$$= 2 \cdot OP \cdot 2\Theta / 2\Pi \cdot OP$$

$$= 2\Theta/\Pi$$

$$= (2/\Pi)\arcsin(a/2h) \qquad (\text{?}.3.1)$$

We can view the above probability either way--

(i) the probability of a point falling within an angle $\Theta$ as stated above

(ii) the probability of a point falling within the orthogonal distance of $+a/2$ to $-a/2$.

The technique involved in generalizing the above probability to higher dimensions, say k, is simple. We have to obtain the surface element of a sphere of radius h in k dimensions and the surface element cut off by the k-dimensional arc which subtends an angle $\Theta$ at the center of the sphere. We shall illustrate the procedure for 3-dimensions before generalizing it to k-dimensions. The 3-dimensional sphere can be obtained by revolving a semicircle around its diameter. The surface element generated by arc between $y = -a/2$ and $y = a/2$ is twice the element generated by arc between $y = 0$ and $y = a/2$ and hence the 3-dimensional surface element between $y = -a/2$ and $y = +a/2$ can be expressed as

$$C_3 = 2 \cdot 2\Pi \int_0^{a/2} x\,ds$$

where $x = \sqrt{h^2 - y^2}$ and $ds = hdy/\sqrt{h^2 - y^2}$

$$\text{Hence } C_3 = 2 \cdot 2\Pi \int_0^{a/2} (h\sqrt{h^2 - y^2})/(\sqrt{h^2 - y^2})\,dy$$

$$= 2\Pi ah$$

and $S_3$, the total surface element will be

$$2 \cdot 2\Pi \int_0^h (h \cdot \sqrt{h^2 - y^2})/(\sqrt{h^2 - y^2})\, dy$$

$$= 4\Pi h^2$$

Hence, $P_3 = C_3/S_3 = a/2h$ (3.3.2)

For k dimensions, the (k-1)-dimensional semisphere is required to be revolved around a (k-2)-dimensional hyperplane. In the above notation, the probability $P_k$ can be expressed as

$$P_k = \frac{2 \cdot \frac{S_{k-1}}{h^{k-2}} \cdot h \cdot \int_0^{a/2} \frac{(\sqrt{h^2 - y^2})^{k-2}}{\sqrt{h^2 - y^2}}\, dy}{2 \cdot \frac{S_{k-1}}{h^{k-2}} \cdot h \cdot \int_0^{h} \frac{(\sqrt{h^2 - y^2})^{k-2}}{\sqrt{h^2 - y^2}}\, dy}$$

$$= \frac{\int_0^{a/2} (\sqrt{h^2 - y^2})^{k-3}\, dy}{\int_0^{h} (\sqrt{h^2 - y^2})^{k-3}\, dy}$$

In the numerator of the above expression, substitute $y = h\sin\phi$. Then, $dy = h\cos\phi d\phi$ and the integral reduces to

$$h^{k-2} \int_0^{\alpha/2} \cos^{k-2}\phi d\phi$$

where $\alpha/2 = \arcsin a/2h$. By the same substitution, the denominator could be reduced to

$$h^{k-2}\int_0^{\Pi/2} \cos^{k-2}\phi d\phi$$

If we let $\sin^2\phi = z$, this integral can be reduced to the complete Beta integral

$$\frac{1}{2}B\left(\frac{1}{2}, \frac{k-1}{2}\right) = \frac{1}{2}\int_0^1 z^{-1/2} (1-z)^{(k-3)/2} dz$$

and the denominator could be reduced to

$$\frac{1}{2}\int_0^{\sin^2\alpha/2} z^{-1/2} (1-z)^{(k-3)/2} dz$$

Therefore, $P_k$, the ratio can be expressed as

$$P_k = B(\sin^2\alpha/2;\ 1/2,\ (k-1)/2) \qquad (3.3.3)$$

where the right side of (3.3.3) is the incomplete Beta function. This is the probability of a point falling within (-a/2, +a/2) in k dimensions. We will now make use of this probability expression in studying the simple structure configuration.

## 3.4 Determination of Simple Structure Configuration

Let us assume that we have a total of n points in p-dimensional space. As stated in the previous section, the probability, in p-dimensional space, of a point falling within (-a/2, a/2) is given by

$$P_p = B(\sin^2\alpha/2;\ \frac{1}{2},\ (p-1)/2) \qquad (3.4.1)$$

where $\alpha/2 = \arcsin a/2h$. Without loss of generality, h could be taken to be 1. We are interested in determining whether these are points lying on a subspace. In a p-dimensional space, (p-1) points always lie on a (p-1)-dimensional subspace. Thus out of a total of n points, we have available only (n-p+1) free points. Likewise, if we are to make any statement about r points lying on a subspace, we can only look to (r-p+1) points as (p-1) of them will always lie on a subspace. If we denote by $P_p$ the probability of a single point lying within a fixed distance ±a/2 from a given (p-1)-dimensional hypersphere, in p-space, the probability that (r-p+1) points will fall in that region out of (n-p+1) is given by

$$\binom{n-p+1}{r-p+1} P^{r-p+1} \; 1-P_p^{n-r}$$

Thus, if a region is to contain more than r points, the probability would be given by the cumulative binomial distribution

$$\sum_{i=r}^{n} \binom{n-p+1}{r-p+1} P_p^{\,i-p+1} \quad (1-P_p)^{n-i} \qquad (3.4.2)$$

Also on the basis of the (n-p+1) free points available, the expected number of points (in excess of (p-1) falling in a (p-1)-dimensional region is

$$(n-p+1)P_p \qquad (3.4.3)$$

The sum of binomial terms (3.4.2) can be evaluated by the incomplete Beta function. The result can be summarized as

$$\begin{aligned} &P[r \text{ or more of } n \text{ points lie within } \pm a2] \\ &= B(P_p;\ r-p+1,\ n-r+1) \qquad (3.4.4) \end{aligned}$$

where $P_p$ itself is an incomplete Beta function. If this probability is very small we shall say that the simple structure subspace determined by the points lying on it is overdetermined. However, the probability of a subspace being overdetermined itself depends on the probability $P_p$. We, therefore, need to consider the interval width and the probabilities, so that the "probability of overdetermination" may be a meaningful concept. Bargmann [2], in studying the overdetermined subspaces in relation to factor analysis fixed the ratio a/h to be ±0·10. These criteria, which reflect common usage in factor analysis, did not suit our requirements.

The object of this test, as will be discussed in Chapter IV, is to suggest to the viewer of a graphics display, some vectors which are normal to overdetermined hyperplanes. If no such concentration were present, the expected number of points, would be, (according to (3.4.3) and the discussion on free points)

$$(n-p+1)P_p + (p-1) \qquad (3.4.5)$$

After considerable experimentation, with n between 20 and 50, and p between 2 and 5, we found that taking $P_p$ such that the expected number of points is $(p+5)$, i.e.,

$$P_p = 6/(n-p+1) \qquad (3.4.6)$$

gave satisfactory results in the identification of overdetermined hyperplanes by the Beta test (3.4.4). A value of $P_p$ smaller than this was too stringent so that a considerable number of points would have to lie on a subspace before it could be considered well determined. The suggested value of $P_p$ was fairly moderate. We compensated for this value by tightening up the probability level for declaring a subspace to be overdetermined. We set this probability at 0.01.

The arbitrariness of choice of $P_p$ and levels of significance may be disquieting to some readers. They may be reminded though, that we are dealing with a phase of data analysis which is exploratory. A sample from a p-variate normal distribution, with a dispersion matrix of rank (p-1), will always be on a (p-1)-dimensional subspace, exactly. There is no test for a hypothesis in the population. Rather, we deal with an instance where, after some linear transformations of the original variables, one of them has negligible variance, after some points have been deleted from the sample. Consequently, a decision as to what is negligible, and what is "close to a plane" is really as arbitrary as declaring that, viewed from some point in the universe, most (but not all) of the stars of a galaxy lie close to a plane. In the final analysis, only the graphic display of certain projections will reveal such intuitive configurations.

In our examples, the subspaces were overdetermined at much smaller values than 0.01 which points to the fact that (3.4.6) was quite useful. For the rest, the choice of critical values is as arbitrary as "0.05 level" (because we have five fingers?) and the 0.01 level of significance. As a guide for displaying configurations, our two levels of $P_p$ and $\alpha$ were "useful."

# CHAPTER IV

# CLUSTERS IN SUBSPACES--IDENTIFICATION

## 4.1 Introduction

In the previous chapter, we addressed ourselves to the statistical aspects of well determined subspaces and derived a few pertinent geometric probability expressions which may guide us to find such spaces. No mention was made of techniques for finding such configurations. In the present chapter, we consider (a) the technique of determining points lying on a subspace; and (b) the problem of which variable could be discarded for the points which describe an overdetermined subspace.

## 4.2 Overdetermined Subspaces

The problem of determining subspaces in our case is rather similar to the determination of simple structures in factor analysis. The essential difference, however, lies in the fact that a factor analyst looks for simple structure among variables. From data points, he constructs a correlation matrix and gets a factor matrix by applying one of the many suitable techniques available for this purpose. If he so desires, after obtaining an initial solution of the factor matrix, he may obtain a "preferred" representation, e.g., Lawley's form [14], etc. For a factor analyst such a solution may not serve his purpose if he is interested in relating the artificial variables to observable ones. In that case he will have to resort to some other forms of representation, such as the simple structure technique. The number of artificial variables required to explain an observable variable is known as the complexity of the observable variable. For the purpose of interpretation of artificial variables, it is desirable that complexities of observable variables be low. Both analytic and geometrical techniques are available to a factor analyst to express the final solution in a form suitable for interpretation.

We have a slightly different problem. First of all, we do not look for any artificial variables to represent the observable variables. Thus whereas a factor analyst works on a factor matrix, we work on the data matrix itself. The starting point for a factor analyst is the correlation matrix obtained from the data matrix. A somewhat similar standardization is employed in our case, except that we standardize the data matrix on the basis of cluster means and the "within" matrix and then normalize the points to unit length. After displaying these points, and the unit ellipses around the cluster means, on all 2-dimensional directions, we proceed to single out those points which could be described in terms of fewer variables. In this connection, it does not concern us how far apart these points are, as long as they lie on a subspace of lower dimensionality. Thus this technique has an advantage that it can identify points lying on a subspace even though they may be belonging to different populations, or clusters. This is precisely what we had in mind. The technique of cluster analysis assigns points to the populations to which they belong. Leaving this structure intact, our new technique determines points which lie on a subspace.

A discussion may be in order regarding the number of subspaces one can find. If we can determine one overdetermined plane, the chances are that there are many planes in the vicinity of a plane already found.

The reason is that, by "tilting" the plane already found, we can still retain many of the points belonging to the original subspace found, pick up a few new points and obtain another overdetermined plane. However, we can reduce the multiplicity of subspaces by ignoring these additional planes found which lie within a certain range of the original plane. This can be done by requiring a minimum angle between the normals to two distinct planes. What angle should be maintained between two planes before declaring them as distinct is a matter of choice. The "orthogonality" preferred by some factor analysts is, at least for our problem, quite useless.

## 4.3 Identification of Subspaces

In this section, the general identification technique will be described. Let there be $n$ experimental units, each with $p$ measurements. The data matrix of order $n \times p$ will be designated as X. This matrix is subjected to a cluster analysis program. If the data are normally distributed, we need to use virtual clusters; hence we used the program developed by Bargmann and Graney [5] for this purpose. After NG (computer program notation) such clusters have been found, we may define a matrix A, with elements $a_{ij}$, of order $n \times NG$ such that

$$\begin{aligned} a_{ij} &= 1 \quad \text{if unit } i \text{ belongs to cluster } j \\ &= 0 \quad \text{otherwise} \qquad (4.3.1) \end{aligned}$$

Let $D_k$ (of order $NG \times NG$) denote a diagonal matrix with elements $k_j$, $j = 1, 2, \ldots, NG$ where $k_j$ is the number of points assigned to cluster $j$. Subtraction of cluster centers or cluster means from each point produces a matrix Y, formally given by,

$$Y = X - AD_k^{-1}A'X \tag{4.3.2}$$

$$= (I - AD_k^{-1}A')X \tag{4.3.3}$$

Now we can obtain an estimate of the common dispersion matrix $\Sigma$, using the within-cluster sample dispersion matrix

$$S = (1/n_e)Y'Y \tag{4.3.4}$$

where $n_e = n-NG$. For the determination of subspaces, we must first transform our reference frame to a Cartesian metric (which is, of course, merely a computational device, and never actually displayed). As a convenient technique, we used the Gram-Schmidt ("Forward Doolittle") reduction,

$$S = TT' \tag{4.3.5}$$

where T is a lower triangular matrix with positive diagonal elements, hence unique. In terms of this Cartesian reference frame, the data matrix is now transformed into

$$Z = Y(T')^{-1} \tag{4.3.6}$$

To find the "simple structure" subspaces, we look for unit vectors $\underline{t}$, such that

$$Z\underline{t} = \underline{v} \tag{4.3.7}$$

and $\underline{v}$ has the property that as many elements as possible are close to zero in the following sense:

Let $\underline{z}_i'$ denote the ith row of Z and $v_i$ denote the ith element of $\underline{v}$. Then it is clear that

$$\underline{z}_i'\underline{t} = v_i \tag{4.3.8}$$

The ith element in $\underline{v}$, namely $v_i$, is considered close to zero if

$$v_i/\sqrt{\underline{z}_i'\underline{z}_i} < \sin \alpha/2 \tag{4.3.9}$$

where $\sin \alpha/2$ is determined by (3.4.1).

Let us now consider the significance of the expressions (4.3.8) and (4.3.9). $\underline{t}$ is a unit vector and so is $\underline{z}_i'/\sqrt{\underline{z}_i'\underline{z}_i}$. Their "inner product," $\underline{z}_i'\underline{t}/\sqrt{\underline{z}_i'\underline{z}_i}$, therefore is the cosine of the angle between the vectors $\underline{t}$ and $\underline{z}_i'$. But

$$\underline{z}_i'\underline{t}/\sqrt{\underline{z}_i'\underline{z}_i} = v_i/\sqrt{\underline{z}_i'\underline{z}_i} \qquad (4.3.10)$$

and hence $v_i/\sqrt{\underline{z}_i'z_i}$ is the cosine of the angle between the vectors $\underline{t}$ and $\underline{z}_i'$. Since $\underline{t}$ is unit normal to the subspace (4.3.9) is established.

For a given number of experimental units, n, and p measurements on each of them, we can determine $P_p$ using (3.4.6) and hence $\sin \alpha/2$ using (3.3.4). (4.3.9) is then a test to determine if the vector corresponding to a given data point (reduced in terms of z's) is close to the subspace to which $\underline{t}$ is orthogonal. If the vector (and hence the data point) is close to the subspace, we regard the point as falling in the overdetermined region, and treat the corresponding element of $\underline{v}$ as "zero." We can examine each element of $\underline{v}$ in this manner and determine how many "zero" elements are there. It is the count of these "zero elements" which we subject to the Beta test (3.4.4). If this test is significant, we say that the subspace is overdetermined and report it as a solution, provided it is not "close" to any subspace already found.

The transformation vectors $\underline{t}$ are found in a manner analogous to Thurstone's "Analytical Method" combined with his earlier "Single Plane Method" [36]. According to this method, each row of the reduced data matrix Z is used as a point of departure to find a vector $\underline{t}$. Let us start with the ith row vector $\underline{z}_i'$. We shall assume that $z_{i1}, z_{i2}, \ldots, z_{ip}$ are elements of $\underline{z}_i'$. Then $\underline{t}_o$, the initial approximation to $\underline{t}$ is obtained by normalizing $\underline{z}_i'$, i.e.,

$$\underline{t}_o' = \underline{z}_i'/\sqrt{\underline{z}_i' \underline{z}_i} = (t_{o1}, t_{o2}, \ldots, t_{op})$$

with this trial, the projections

$$\underline{v}_o = Z\underline{t}_o$$

are calculated and the elements $v_{oi}$ are tested for closeness to zero. If a value is very close, a large weight is assigned to the point, for the subsequent weighted regression technique. If it is large, the weight may be zero. Following Thurstone, we use discrete (step) weights, as follows:

If $0 < v_{oi}/\sqrt{\underline{z}_i' \underline{z}_i} < \sin \alpha/2$

$w_i$ = BND (BND, bound, initially 8).

If $\sin \alpha/2 < v_{oi}/\sqrt{\underline{z}_i' \underline{z}_i} < 2\sin \alpha/2$

$w_i$ = BND-1

If $v_{oi}/\sqrt{\underline{z}_i' \underline{z}_i} >$ BND x $\sin \alpha/2$

$w_i = 0.$

With these weights we can follow a weighted regression scheme to obtain an improved vector $\underline{t}_1$. This can be further simplified by a relation (due to Thurstone) which expresses

$$t_{1j}^* = \frac{t_{oj} - u_j}{1 - t_{oj}u_j} \qquad (4.3.11)$$

$$\text{where } u_j = \sum_{i=1}^{n} z_{ij}v_{oi}\sqrt{w_i} \Big/ \sum_{i=1}^{n} z_{ij}^2\sqrt{w_i} \quad \ldots \qquad (4.3.12)$$

$t_{1j} = t_{1j}^*/\sqrt{\underline{t}_1^{*\prime}\underline{t}_1^*}$ are then

elements of the new trial vector $\underline{t}_1$. We calculate

$$\underline{v}_1 = Z\underline{t}_1$$

and once again the weights are assigned following the scheme detailed above.

However, before assigning these new weights the BND value is reduced to 3, in the next cycle to 2 and then kept at 1 for the remaining iterations. Beginning with $\underline{t}_0$, seven iterations are performed which give rise to $\underline{t}_1, \underline{t}_2, \ldots, \underline{t}_7$. The last one, $\underline{t}_7$, is retained as $\underline{t}$ and $Z\underline{t} = \underline{v}$ is formed. If the number of "zero" entries in this $\underline{v}$ is significantly large as explained earlier, we will have determined a well defined subspace, to which $\underline{t}$ is normal. The entire process is repeated, with each row taken as a trial value. A given row may or may not identify a subspace. If it leads to an overdetermined subspace, the solution, except for the first one, is checked as explained in section 4.2 to ensure that the subspace is "different" from any of the ones already found from previous rows. For this purpose, we require that the cosine of the angle between two planes should not exceed 0.7 meaning that the planes were apart by at least 45°. Once again we would like to mention that this is an arbitrary requirement; we found this useful in our experimental studies.

Now we must retransform to the original data frame. The vectors $\underline{t}$ that we obtain above, are with reference to the matrix Z. Our original objective was to remove mean shifts and then look for experimental units which lie on subspaces. However, in working with Z, we have removed the mean shifts and also reduced the metric to an orthogonal Cartesian frame. This was a matter of convenience. To restore the original metric, we must obtain the solution in terms of Y. This can be easily obtained as under. (4.3.6) is the transformation of Y into Z and (4.3.7) is the simple structure solution in terms of Z. Substituting (4.3.6) into (4.3.7) we obtain

$$Y(T')^{-1}\underline{t} = \underline{v} \qquad (4.3.13)$$

from which we conclude that $(T')^{-1}\underline{t}$ is the transformation in terms of Y.

The computer program reports both $\underline{t}$ and $(T')^{-1}\underline{t}$. The vector $\underline{t}$ is reported under the heading "Vector which transforms original factor matrix into the above plane no. v_" and $(T')^{-1}\underline{t}$ is reported under the heading "Transformation vector to transfer raw data to simple structure." The corresponding elements of $\underline{v}$ are also reported. We shall also refer to elements of $\underline{v}$ as the "loadings" or "scores" of original points with reference to the simple structure plane determined.

For each $\underline{t}$, only the vectors $(T')^{-1}\underline{t}$ are conveyed to the display program, since we plot the original data points, in terms of the observed variables. As well be explained in Chapter 5, the display program takes one of the observed variables as one axis and some specified vector as another axis. For a given choice of a variable and a vector, the vector is reduced in such a manner that it forms an orthogonal frame of reference with the chosen observed variable. A question may be raised as to why one of the axes always corresponds to an observed variable. In this connection, it should be pointed out that a vector specified by the user, is equivalent to an artificial variable, a linear combination of the observable ones. The elements of the vector given by the user serve as weights in forming this linear combination. When we view the displays with reference to an orthogonal frame of reference consisting of an observable variable against an artificial variable, we can make an inference as to how an observable variable compares with an artificial variable. If both reference axes were to correspond to artificial variables, the displays would be hard to interpret in a realistic sense. This is our main consideration in insisting that one of the axes correspond to an observable variable. Further, if we should permit a user to select any two vectors, these could be translated into an orthogonal frame of reference in many different ways leading to utter confusion.

## 4.4 Elimination of Variables

One notable difference between the rotational problem in factor analysis, and the problem presented in this dissertation, is the fact that the former focuses attention on those points (variables, in that case) which are far removed from the subspaces. By contrast, the latter pays attention to only those points which are so close to a subspace that they define it. It is these data points only which require one variable less for adequate description. It should be noted, again, that such a subspace is not a single region within the p-dimensional space. Rather, for each simple structure plane, there are as many (parallel) (p-1)-dimensional hyperplanes as there are clusters. Points which, in this sense, fall into the same subspace may be far removed from each other, inasmuch as they may be in different clusters. But even with their distinct neighbors, they share the property that the same (p-1) variables are sufficient to explain their characteristics. It is for this reason that we expect entirely new principles of classification of data points, different from what could be expected by varying or refining cluster analysis or factor analysis techniques.

Mathematically, we could use for description, (p-1) linear combinations of the original p variables, with the combinations chosen within the hyperplane orthogonal to the $(T^{-1})'\underline{t}$ vector. For real life interpretation, this procedure would be useless. Surely we are better off with p observable variables than with (p-1) artificial ones. The principle of parsimony is not just a principle restricted to dimensionality. It would seem reasonable, then, to eliminate, for the data points close to one of the subspaces, that observable variable which contributes the least to the description of this selection of data points.

As a measure of proximity of each variable to the artificial variable which defines the subspace, we propose to use the correlation between the original variables and the artificial variable. This can be obtained as follows. Recall that Y is obtained from the original data matrix after subtraction of the appropriate cluster means. Hence

$$\underline{1}'Y = \underline{0}' \qquad (4.4.1)$$

where $\underline{1}'$ is a row vector consisting of all 1's and $\underline{0}'$ is a null row vector. Also,

$$\begin{aligned} \underline{1}'\underline{v} &= \underline{1}'Z\underline{t} \\ &= \underline{1}'Y(T')^{-1}\underline{t} && \text{(By (4.3.6))} \\ &= \underline{0}' && (4.4.2) \end{aligned}$$

and

$$\begin{aligned} \underline{v}'\underline{v} &= \underline{t}'Z'Z\underline{t} \\ &= n_e\underline{t}'I\underline{t} \\ &= n_e && (4.4.3) \end{aligned}$$

since $\underline{t}$ is a unit vector. By virtue of the fact that $Z'Z = n_eI$, and because of (4.4.1) $(1/n_e)Y'\underline{v}$ will be an unbiased estimate of the covariances between the original variables and an artificial one on which the data points have scores which are the elements of $\underline{v}$. There will be as many $\underline{v}$ vectors as there are overdetermined subspaces (each corresponding to a different, but possibly overlapping, selection of data points). For each of these, we must determine its correlations with the original variables. Thus, if there are 4 variables and 3 different solutions (overdetermined subspaces), we will have a total of 4 x 3 = 12 correlations. Now, for a given $\underline{v}$ (a given subspace),

$$\begin{aligned} Y'\underline{v} &= Y'Z\underline{t} \\ &= Y'Y(T')^{-1}\underline{t} && \text{(By (4.3.6))} \\ &= n_e TT'(T')^{-1}\underline{t} && \text{(By (4.3.4) and (4.3.5))} \\ &= n_e T\underline{t} && (4.4.4) \end{aligned}$$

T is the same lower triangular matrix that was used in reducing the metric underlying the matrix Y to a Cartesian reference frame. From (4.4.4), we conclude that

$$(1/n_e)Y'\underline{v} = T\underline{t} \qquad (4.4.5)$$

Hence the estimated covariances between the original variables and a single artificial variable, are elements of $T\underline{t}$. To obtain the correlations, we have to divide each of these elements by the square root of the estimate of the variance of the artificial variable and the square root of the estimate of the variance of the observable variable. Using (4.4.2) and (4.4.3), we conclude that the estimate of the variance of the artificial variable is

$$(1/n_e)\underline{v}'\underline{v} = 1 \qquad (4.4.6)$$

since a sum of squares ($\underline{v}'\underline{v}$) must be divided by degrees of freedom to produce a variance estimate. Hence to obtain the correlations, we have to divide the elements of $T\underline{t}$ by the square root of the estimate of the variance of the observable variable only. In the computer program, after the vector $\underline{t}$ is obtained, the product $T\underline{t}$ is formed and the correlations are then calculated by division of each of the elements of $T\underline{t}$ by the square root of the estimate of the variance of the corresponding observable variable. The estimates of the variances of the observables are still available in the final step of the cluster analysis part of the program and these values are stored for use at this time. T is also stored.

Once the correlations between the original variables and their scores with reference to a subspace (i.e., vector $\underline{v}$) are available, it is easy to determine which variable should be discarded. In discriminant analysis, we come across the concept of correlations between original variables and the discriminant function. The discriminant function is nothing but a linear combination of the original variables which discriminates best between experimental units in a specified sense. The purpose for which we want to discriminate is important as the discriminant functions for different purposes are usually different. Given these things, the variable which correlates most strongly in absolute value with the discriminant function is the most important in discriminating. In our study the vector $\underline{t}$ which transforms the original observations into $\underline{v}$ plays a similar role in the sense that the elements of a well defined subspace represented by $\underline{v}$ has most elements near zero (see section 4.3) and a few far removed from zero. In analogy with discriminant analysis, $\underline{v}$ is the vector that produces the two groups of data points. Thus, the variable which correlates most strongly with this artificial variable, contributes most to the discrimination process. It is this maximally correlated observed variable which should be eliminated for, after it has been discarded the other variables contribute far less to the discrimination between these two sets of data points. If only one variable is to be sought which would explain the difference between these data points which fall into the subspace and those that do not, it would be this one which is closest (has highest correlation in absolute value) to the expendable artificial variable. Note the exact opposite of this technique to discriminant analysis, where we seek the best discriminator. Here we identify the "most expendable" artificial variable. By our correlational technique, we have identified, for each subspace, that observable variable which is closest to the artificial variable <u>not</u> needed in the description of the selected data subset. Note again, the need for this correlational interpretation. If it were argued that the artificial variable itself ought to be discarded, we would be left with (p-1) <u>artificial</u> variables, a rather unsatisfactory situation. After the correlational approach, we have (p-1) <u>observed</u> variables left.

# CHAPTER V

# DESCRIPTIONS OF COMPUTER PROGRAMS

## 5.1 The Computer Programs

The algorithms explained in previous chapters have been synthesized into two computer programs which are available at the University of Georgia. The first program, named CLUSTR, and described in the next section, identifies the clusters and overdetermined subspaces. The second one is available as a conversational system for the IBM 2250 Graphics unit. In this chapter, we describe these two computer programs. The following chapter will contain instructions regarding the use of these programs, and the interpretation of graphical displays.

## 5.2 An Algorithm for Identification of Points Lying on a Subspace

In this program, beginning with the data matrix, we first identify the clusters. This is essentially the algorithm proposed by Bargmann and Graney [5]. However, the algorithm proposed by Bargmann and Graney stops at the identification process. Since we also need to identify points lying on an overdetermined subspace, we extend the algorithm further. A complete listing of the program is contained in Appendix A. This program can be logically divided into 2 parts. Up to statement number 811, it is essentially the reproduction of the program developed by Bargmann and Graney [5], where a complete documentation of this part can be found. We have made a small change in the program to suit our needs. As explained in Bargmann and Graney [5], their program makes three passes to identify clusters. The passes made in their program are controlled by the statement immediately following statement number 444. In our program, CLUSTR, this has been replaced by the transition to the subspace-identification program. Further, the program developed by Bargmann and Graney [ 5 ] does not calculate cluster means or the "within" matrix after three passes. Their program computes these quantities at the beginning of the first pass, and after the first and second passes. We require these quantities for our search for the points lying on subspaces. These quantities are calculated in statements between number 811 and number 20001. At this stage, we also punch cards containing the means for each cluster and each variable. These will be needed prior to the execution of ELLIPSE described below. In statements between number 11020 and number 10001, we standardize the original points and reduce them to zero mean and a Cartesian metric. This, then, is the beginning of the second logical part of the program. Here, we determine the points lying on subspaces, using the method of weighted least squares together with Thurstone's "Analytical Method" and the "Single Plane Method." The algorithms (including the change of the BND variable) have been presented in sections 4.2 and 4.3

The output of this program consists of two parts--a print out and a punched deck. The printed output contains the results of three passes made to find clusters. The results listed after the third pass are the final results relating to cluster analysis. It shows which point belongs to which cluster. It also shows at what level the points got included in the cluster. The second part of the printed output gives various simple structure solutions. It gives vectors which transform the original observations into simple structure planes. For each of these vectors, the correlations between the original variables and the "scores" of points with reference to this vector, the number of points falling into the simple structure plane corresponding to this vector, and the probability of these many points falling into this simple structure plane, are given. A simple structure plane is not included as a solution if the probability of the number of points falling into this plane is greater than 0.01 or if it is within 45° of a plane already found.

Apart from the printed output, a punched deck is also produced. These are data cards which are later loaded into data sets, as described below. The first few cards in this deck--equal in number to the number of clusters formed--are the cards containing cluster means. The next set of cards contains vectors which transform the original data points into simple structure solutions. The number of clusters is designated by NG and the number of simple structure solutions is designated by NSOL. This program also reproduces the cards for each data point with the following additional information. For each data point, the card contains a serial number in columns 1-3, the number of the cluster to which it belongs in column 4, and the simple structure planes in which it is included in columns 5-14. In column 4, a '1' is punched if the point belongs to cluster number 1, '2' if the point belongs to cluster number 2, etc. '0' is punched in column 4 if the point was not assignable to any of the clusters. The simple structure planes to which the point belongs is indicated in columns 5-14 as follows: A '1' in column 5 indicates that the point falls into simple structure number 1, a '1' in column 6 indicates that the point belongs to simple structure number 2, etc. (with provision for up to 10 solutions). This is certainly more than adequate capability. Zeros or blanks in columns 5 to 14 (the computer program punches zeros) indicate absence of this point in the corresponding simple structure plane. Columns 15-70 contain the original coordinates of each data point. The entire punched deck output is also printed out. The cluster means are printed at the end of the third pass, and the vectors which transform the raw data into simple structure solutions are printed immediately after the printout of the corresponding solution. The information punched into the last NP cards of the punched deck is also printed as a final summary. The user may find it helpful to keep this summary with him while he studies the displays.

## 5.3 Loading the Data Set (Utility Program IEBGENER)

After the user has subjected his data to the CLUSTR program, he should next run the IEBGENER program. This program supplies the output of CLUSTR program as input to the ELLIPSE program. A header card, containing the number of points, the number of variables, the number of groups, and the number of overdetermined subspaces identified by the CLUSTR program, is put before the punched deck produced by the CLUSTR program, and the IEBGENER program is executed. A sample deck set-up for the IEBGENER program is given in Chapter VI; this is a utility routine which transfers the cards to disk.

## 5.4 The Display Program and the Conversational System

The second program of the package serves to display projections of the clusters and subspaces, on an IBM 2250 Graphics Display Unit. It enables a user, at the console, to communicate with the system and to manipulate displays appearing on the scope. The program, named ELLIPSE, is capable of handling up to 8 variables, 50 data points, 10 simple structure solutions and 5 groups or clusters. The numbering of the variables is implicit. The first coordinate, for each data point, is regarded as variable number 1, the second coordinate as variable number 2, etc.

The program is an interactive one in the sense that the user, at the console, can decide on variations for later displays on the basis of what he saw in the earlier ones. The input of the data is so formatted and programmed that, if a user wishes to reassign a point from one cluster to another, or if he wishes to change cluster centers, etc., he only needs to make a change to this effect in the corresponding data cards. This capability gives the user an opportunity to redefine clusters and subspaces on the basis of the displays generated. The interaction between the user and the system is achieved through the use of the programmed function keys and the alphameric keyboard which is a part of the 2250 Graphics Display Unit.

The program includes a main program and 8 subroutines. The main program calls two subroutines CALC and EXIBIT. CALC reads the entire input into the ELLIPSE program. The input consists of a header card containing the number of points, the number of variables, the number of clusters and the number of simple structure solutions; cluster means for each variable; vectors which transform the original data points (raw data) into various simple structure solutions; and the data points, together with information regarding the cluster to which a data point belongs and whether or not it lies on a given overdetermined subspace. As explained earlier, this input is given to the program through the execution of the IEBGENER Utility routine. The first (executable) statement of the CALC subroutine reads the header card. Following this, the DO 14 loop reads cluster means, the DO 114 loop reads the transformation vectors and the

DO 10 loop reads the data points. The array IG is used to store information regarding the cluster to which a data point belongs. Recall that for each data point, columns 5-14 contained '1' to indicate the corresponding simple structure planes to which this point belongs. This configuration of '1's and '0's is read as a 10 digit integer number and stored in the first column of the two-dimensional array INNSOL. The serial number of a data point is stored in the second column of INNSOL. In the DO 422 loop, the array IG is examined to determine the size of each cluster. The DO 429 loop, then, determines the total number of points assigned to clusters. The subroutine COR2 is now called which yields the within sum of squares and products matrix based on all the points belonging to clusters. The upper triangular part of this symmetric matrix is stored, columnwise, in the array DSPROD. In the DO 434 loop, each element of the array DSPROD is divided by $n_e$, the degrees of freedom, to yield an estimate S of the common variance-covariance matrix, $\Sigma$. Immediately following the statement number 434, SINV, a sub-routine from the IBM Scientific Subroutine Package, is called to invert the matrix S. The inverse of this matrix S, stored as the first column of the two-dimensional array A, is the metric (based on all points belonging to clusters) employed for unit ellipsoids around the clusters. The subroutnien CALC then calculates metrics corresponding to overdetermined subspaces. For each of the overdetermined subspaces, the DO 426 loop calculates a metric corresponding to each of the overdetermined subspaces on the basis of the points belonging to it. The inner DO 427 loop examines the 10 digit integer numbers stored in the first column of the array INNSOL to determine the points lying on a particular subspace. <u>For each subspace</u>, the subroutine

CORIS is utilized to calculate a new within-cluster sum of squares and products matrix. In the DO 442 loop, each element of this matrix is divided by the number of points belonging to the overdetermined subspace, to yield an estimate of the variance-covariance matrix based on the points belonging to the subspace only. The subroutine SINV is then called to invert this matrix. The inverse matrix is used as the metric for ellipsoids corresponding to the overdetermined subspace. The metric corresponding to the overdetermined subspace number 1 is stored as the second column of the two-dimensional array A, the metric corresponding to the overdetermined subspace number 2 is stored as the third column of the array A, etc. This is accomplished in the DO 433 loop. The DO 15 loop, beginning at statement number 450, calculates the maximum and minimum values for each variable. These maximum and minimum values are required later for scaling purposes to accommodate each of the projected data points within the screen limits. A flow chart for the subroutine CALC is given in figure 5.4.1.

## Subroutine EXIBIT

The subroutine EXIBIT begins with a call to the DISPLA subroutine of the GRAF (Graphics Addition To Fortran) package [16]. This results in setting up GDSX, GDSY, GTEXT, GPOINT, GDSE, GDSER and GINPUT as display variables. The subroutine LIGHTS of the GRAF package is next called to turn on the lights corresponding to the programmed function keys numbered 1 up to the number of variables, keys 27 to 29 and 31. After these preliminaries, the subroutine MESSGE is called to display an informative message about the program, for the benefit of the user. The subroutine MESSGE returns the control to the subroutine EXIBIT as soon as the user

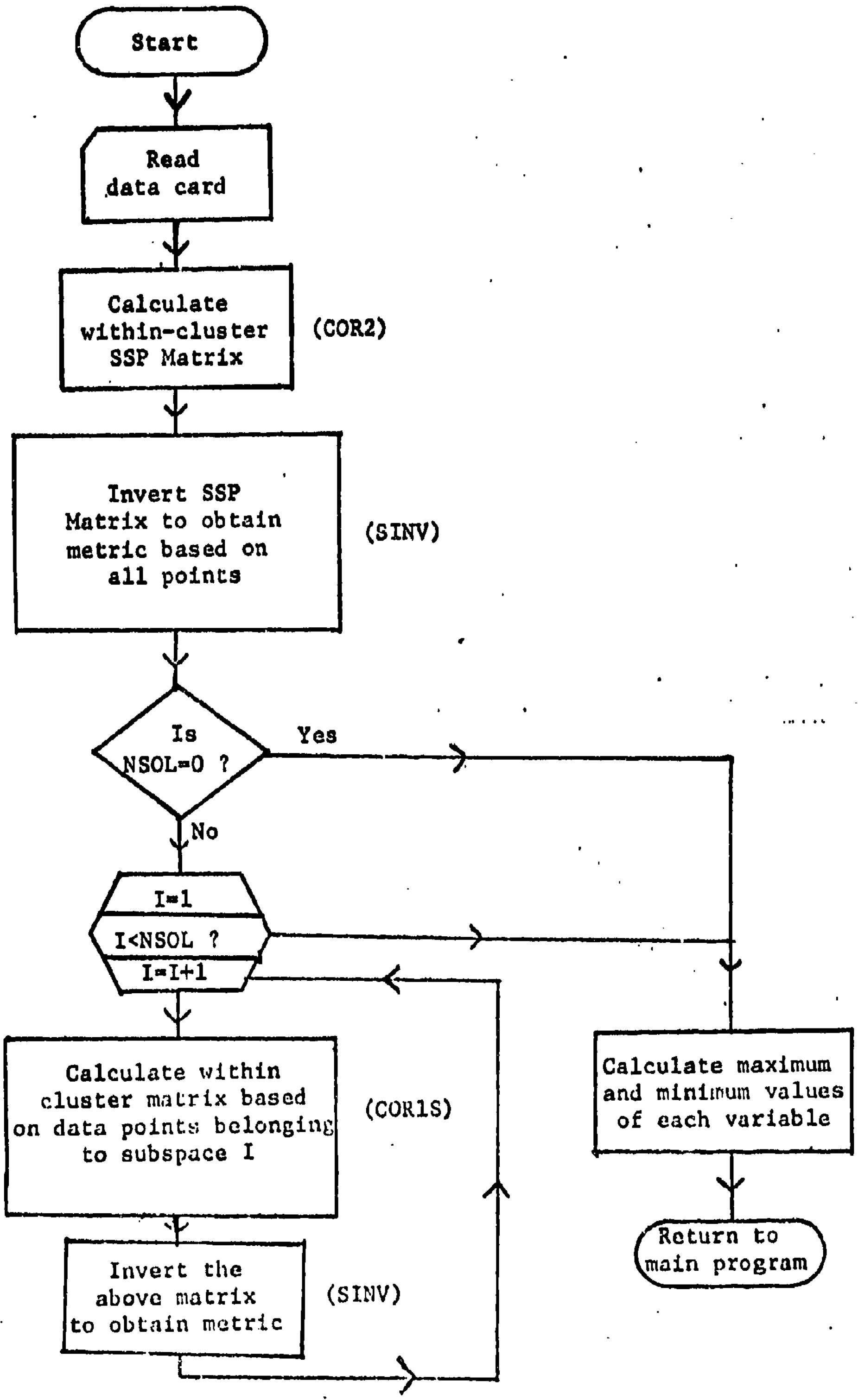
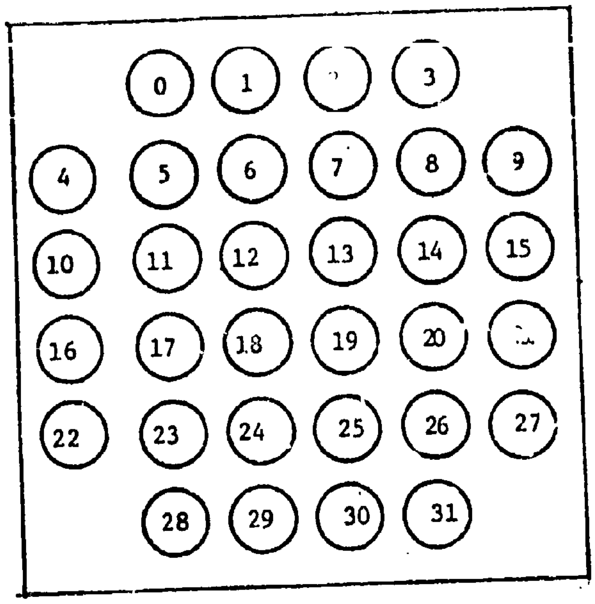
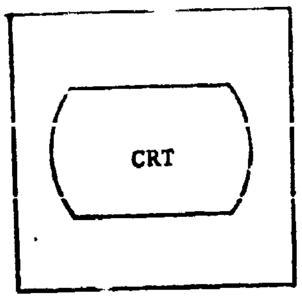

Start
Read data card
Calculate within-cluster SSP Matrix
(COR2)
Invert SSP Matrix to obtain metric based on all points
(SINV)
Is NSOL=0 ?
Yes
No
I=1
I<NSOL ?
I=I+1
Calculate within cluster matrix based on data points belonging to subspace I
(COR1S)
Invert the above matrix to obtain metric
(SINV)
Calculate maximum and minimum values of each variable
Return to main program

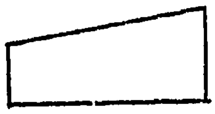
Figure 5.4.1

presses any programmed function key. The message appearing on the screen is erased, a variable NTEST (used later to register the depressed key) is set initially equal to 1 and a variable NGG (a flag which is used to indicate change of vectors) is set equal to 0 and the control goes to statement number 20. This statement is a call to the subroutine KEYIN, with NTEST, the input argument having been set equal to 1. The subroutine KEYIN accepts from the user a vector and the number of a variable, in order to form a 2-dimensional plane. The user indicates his choice of the number of the variable by pressing the corresponding programmed function key. The variable NAXIS is used as an output argument of the subroutine KEYIN and on return contains the number of the programmed function key pressed by the user. As will be seen later, on return from KEYIN, the variable NAXIS must either have a value equal to the number of the variable the user wishes to utilize, or 30. If NAXIS equals 30, it is implied that the user wishes to stop and the display program comes to an end. Otherwise, the subroutine ELLPSE is called with the current value of NAXIS (the number of the variable) as input argument. The subroutine ELLPSE displays the projections of the original data points, and the unit ellipsoids having the metric based on all the points belonging to clusters, onto the 2-dimensional plane formed by the vector and the variable supplied by the user, provided there is no singularity or redundancy involved (see ELLPSE below). After the above displays appear, the user is expected to press a programmed function key. If he presses key 29 or 31, the control comes to statement number 80. This results in erasing the current displays and then a call to the subroutine KEYIN, with NTEST, the input argument, being 29 or 31. As before, the call to the subroutine KEYIN is followed by a call to the subroutine ELLPSE and the cycle



Programmed Function Keyboard





Alphameric Keyboard

IBM 2250 Display Unit

continues. If a key corresponding to the number of an overdetermined subspace was pressed, the control comes to statement number 61, and if key 30 was pressed, the display program comes to an end. If the control comes to statement number 61, the subroutine RELPSE is called to display the projections of the overdetermined subspace. If none of the above mentioned keys is pressed, an error message, as contained in the format statement number 62, appears on the screen. The error message continues to appear until the user presses a proper key or, in case of singular or redundant situations, rectifies the situation. If the subroutine RELPSE is called, after the projections of the overdetermined subspace are displayed, the user is expected to press a programmed function key. Once again, if key 29 or 31 is pressed, the current displays are erased, the control comes to statement number 20 and the subroutine KEYIN is called. The program terminates if key 30 was pressed. If the key corresponding to the number of the overdetermined subspace was pressed, that part of the current display pertaining to the projection of the overdetermined subspace is erased, and the subroutine RELPSE is called to display the projections of the overdetermined subspace that is now being requested. An error message appears if none of the abovementioned keys is pressed, and the cycle continues in this manner. A flowchart of the subroutine is given in figure 5.4.2.

Subroutine KEYIN

The subroutine KEYIN accepts a vector and the number of the variable from the user, to form a 2-dimensional plane. It has one input argument, NTEST, and an output argument, NAXIS. The first statement in the subroutine tests the value of NTEST. If it equals 31, that part of

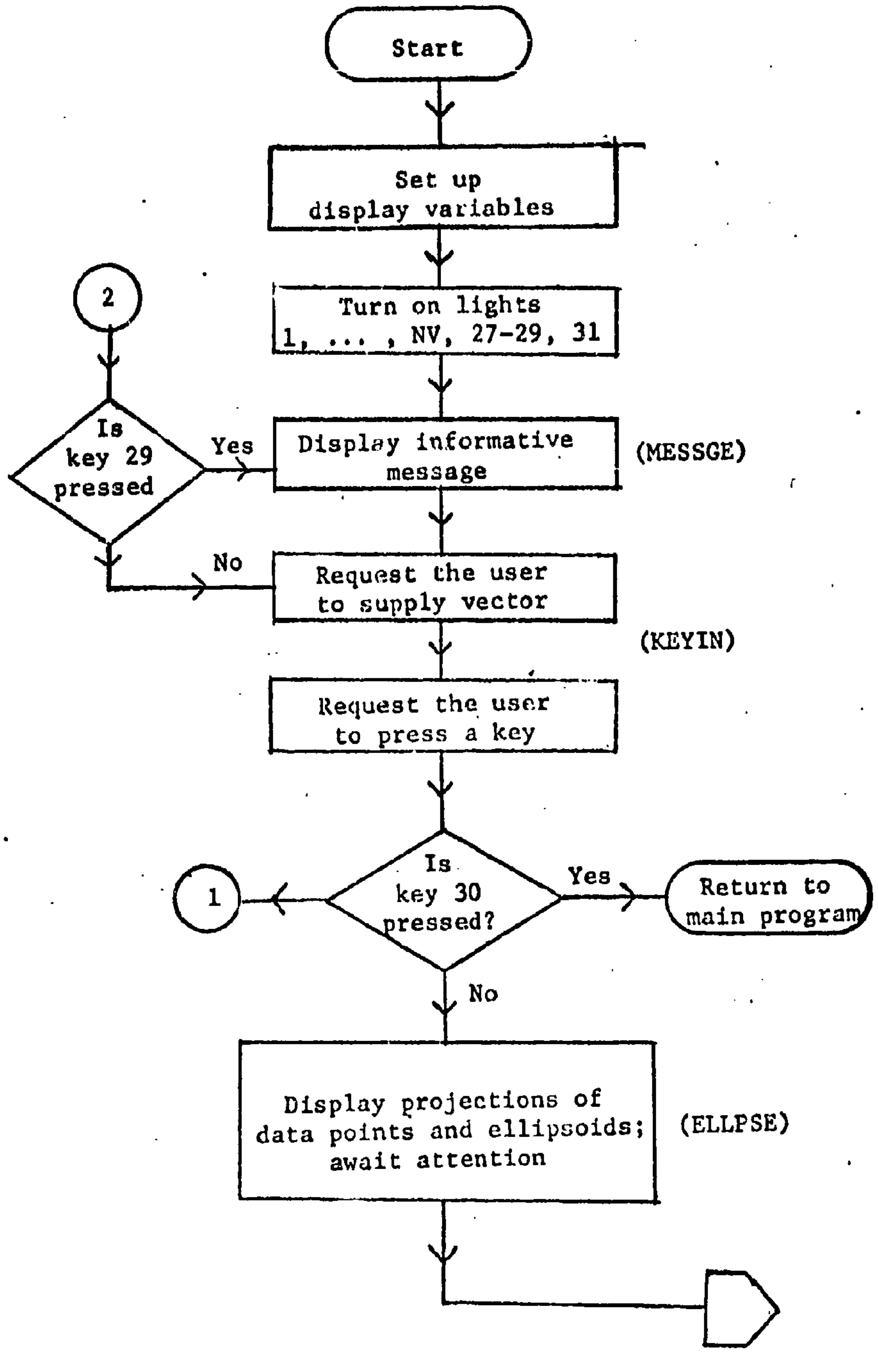

Start
Set up display variables
2
Turn on lights 1, ... , NV, 27-29, 31
Is key 29 pressed
Yes
Display informative message
(MESSGE)
No
Request the user to supply vector
(KEYIN)
Request the user to press a key
Is key 30 pressed?
1
Yes
Return to main program
No
Display projections of data points and ellipsoids; await attention
(ELLPSE)

Figure 5.4.2a

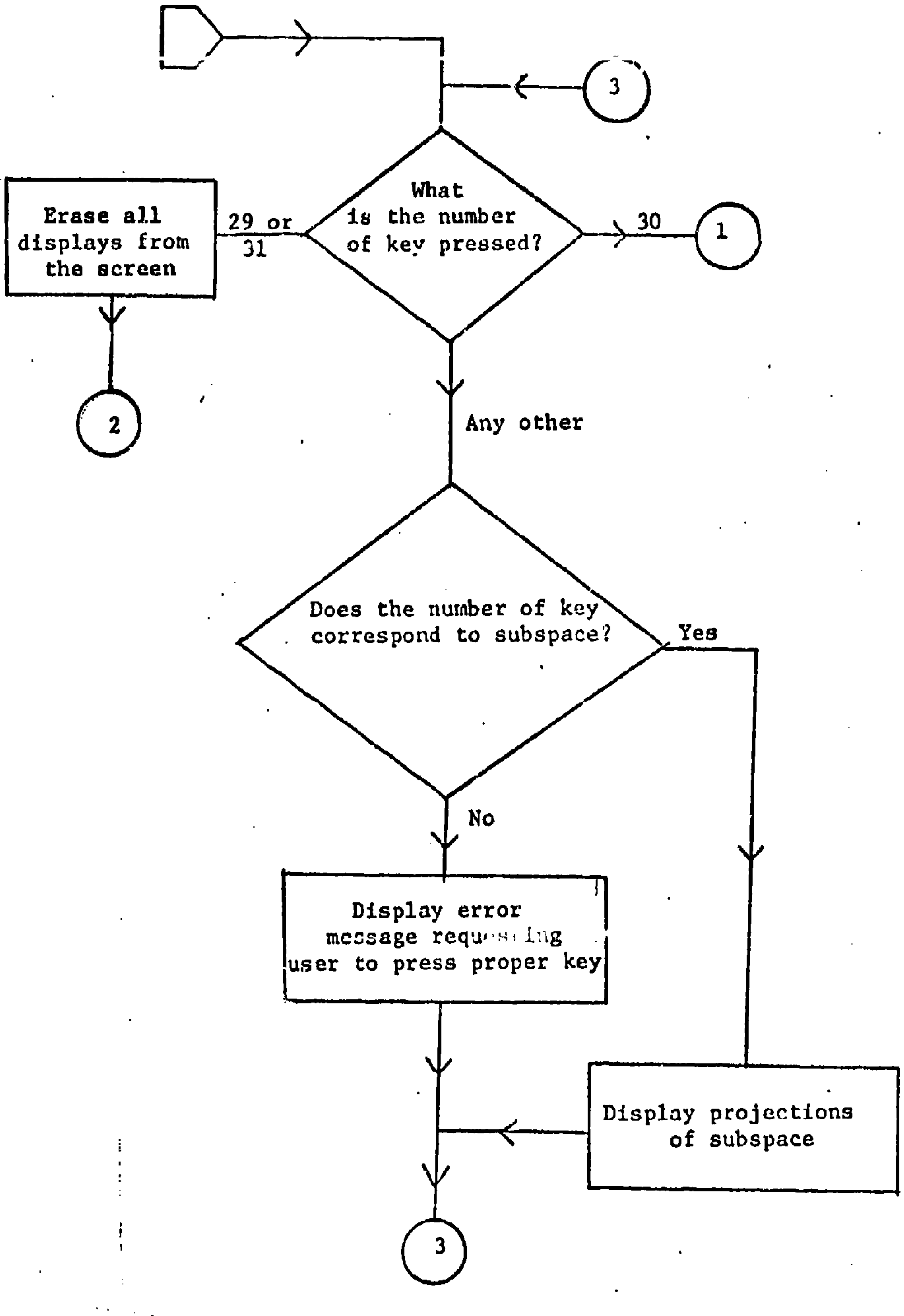

3
What is the number of key pressed?
29 or 31
30
1
Erase all displays from the screen
2
Any other
Does the number of key correspond to subspace?
Yes
No
Display error message requesting user to press proper key
Display projections of subspace
3

Figure 5.4.2b

the subroutine which accepts a vector from the user is skipped, and the control comes to statement number 101. Otherwise, the control comes to statement number 99, and whatever appears on the screen is erased to prepare to accept a vector from the user. On the first call to the subroutine KEYIN, the input argument, NTEST, is set equal to 1 so that the control invariably comes to statement 99. On subsequent occasions, however, NTEST will have a value of 29 or 31. The DO 110 loop displays the transformation vectors suggested by the rotation (CLUSTR) program for the information of the user. The message contained in the format statement number 3000, requesting the user to supply a vector, then appears on the screen. The program now awaits the user to supply the vector. The calls to SCTDV and DVTDM subroutines of the GRAF package transfer the vector supplied by the user from the screen to the display variable table and from there to the dummy unit 4. The vector is read from the dummy unit 4 into the array RINPUT. Before, however, the vector is read, the DO211 loop transfers the current values stored in the RINPUT array to the RWRKNG array. This is done to insure that the vector previously supplied by the user is not destroyed in case he wants to continue with that vector. The DO 213 loop checks if the user supplied a null vector. If so (as is the case when he just wants to continue with the previous vector), this is always replaced by the previous non-null vector as would be apparent from the DO 216 loop. The only exception, as will be seen from the statement following the statement number 213, is the first call to the subroutine. This would result in singularity and the user will receive an error message instead of the displays.

After the vector is accepted, the control comes to statement number 101. The message contained in the format statement number 1658 appears on the screen. The loop beginning with the statement number 60 insures that no value other than 30, or the number corresponding to the variable which the user wishes to utilize, will be returned as the value of the output argument NAXIS. The user can, of course, go back to statement 99, the beginning of the program, if he presses key 28. This gives him a chance to amend the vector already supplied. A flowchart of the subroutine is given in figure 5.4.3.

Subroutine ELLPSE

This subroutine is used to display projections of original data points, and the unit ellipsoids having the metric based on all points belonging to clusters, onto the 2-dimensional plane formed by the vector and the variable supplied by the user. The DO 10 and DO 11 loops set up a matrix R formally given by

$$R = \begin{bmatrix} 0 & a_1 \\ 0 & a_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 0 & a_p \end{bmatrix} \qquad (5.4.1)$$

where 1 in the first column appears in the row corresponding to the number of the variable chosen by the user, and $(a_1, a_2, \ldots, 0, \ldots, a_p)$ is the vector supplied by the user and modified to form an orthogonal axis

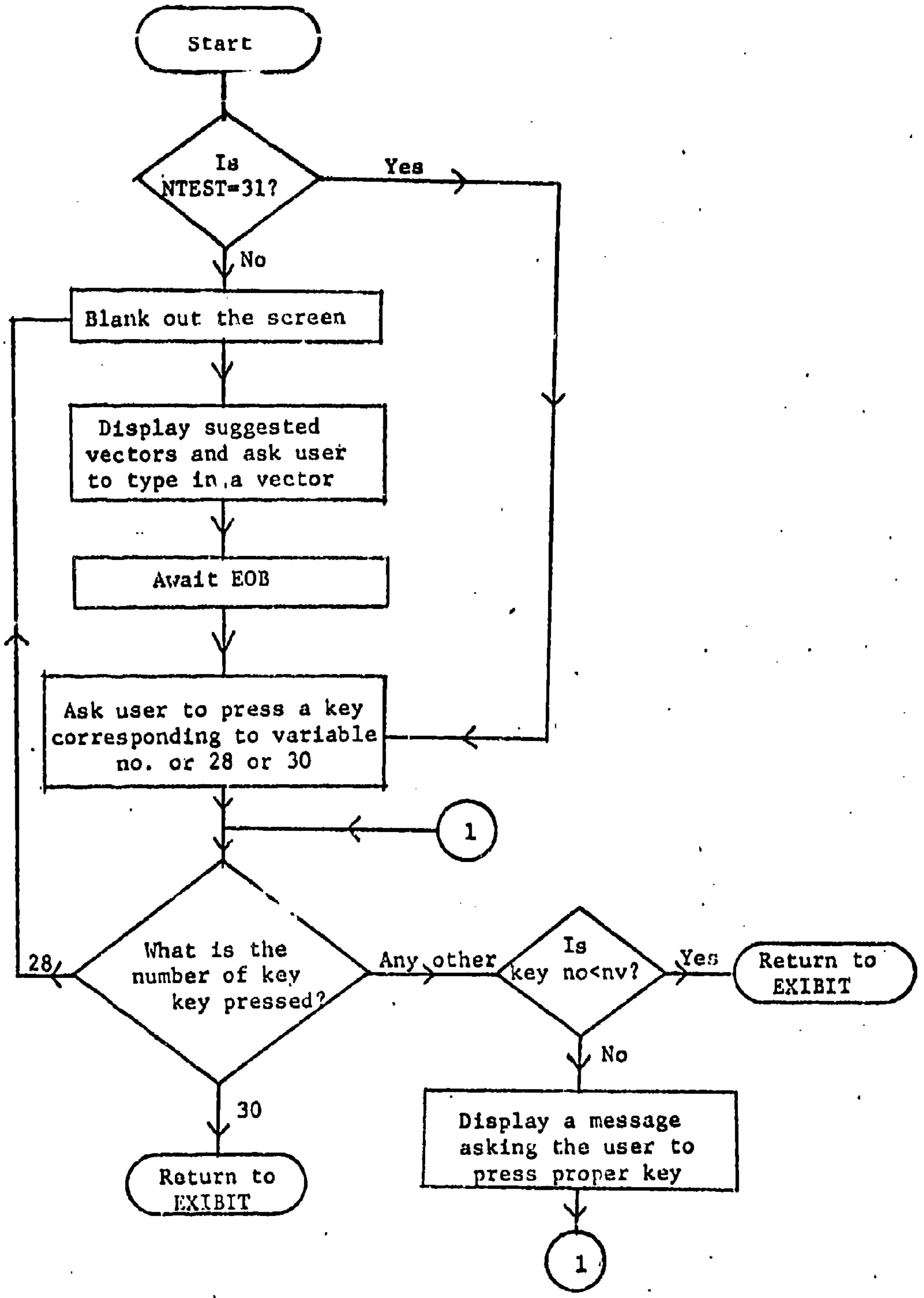

Start
Is NTEST=31?
Yes
No
Blank out the screen
Display suggested vectors and ask user to type in a vector
Await EOB
Ask user to press a key corresponding to variable no. or 28 or 30
1
What is the number of key key pressed?
28
Any other
Is key no<nv?
Yes
Return to EXIBIT
No
30
Return to EXIBIT
Display a message asking the user to press proper key
1

Figure 5.4.3

with the desired variable. If the user supplies a vector collinear with his choice of the observed variable, the second column of the matrix R will have all zero elements. The DO 20 loop, and the statement immediately following this loop, check for the above-mentioned possibility of singularity. If no singularity is present, the control comes to statement number 211; otherwise, the error message, as contained in the format statement number 1659, is displayed on the screen, and control is returned to the subroutine EXIBIT. At statement 211, the DO 21 loop is set up to normalize the second column of the matrix R. The DO 12 loop picks up the metric $S^{-1}$ based on all points belonging to clusters and the subroutines MPRD and GTPRD of the IBM Scientific Subroutine Package are called, to form the matrix product $R'S^{-1}R$. The DO 13 loop, and the statement immediately following it, calculate the matrix product XR, where X is the matrix of original data points. For a projected data point, the element of the first column of the matrix is treated as its x-coordinate and the element of the second column is treated as y-coordinate. The DO 110 loop creates orders to plot points with these sets of x and y coordinates. The actual plotting is done by displaying, on the screen, at the place where the point should appear, its serial number, so that the user may know which data point projects into what region of the 2-dimensional display.

In the statements immediately following the DO 110 loop, the semi-axes of the ellipses to be displayed are calculated. They are based on the metric $R'S^{-1}R$ (of the projected ellipsoids). The cluster centers (centers of ellipses) are likewise transformed into $\underline{\mu}'_i R$. The points describing the circumference of the ellipses

$$(\underline{x}' - \mu_i R)\ R'S^{-1}R\ (\underline{x} - R'\mu_i) = 1 \qquad (5.4.2)$$

where $\underline{x}' = (x_1, x_2)$ are the running coordinates, are constructed by angular sweep. Beginning with an initial angle of 5° the DO 150 loop calculates 72 different points describing the circumference of the ellipses. The DO 100 loop creates orders to plot these points and the ellipses appear on the screen when the statement immediately following the DO 100 loop is executed. A flowchart of the subroutine is given in figure 5.4.4

Subroutine RELPSE

This subroutine is used to display projections of ellipsoids having metric based on the points belonging to the overdetermined subspace being superimposed. If the metric for the ith subspace is denoted by $S_i^{-1}$, the subroutine calculates the points describing the circumference of ellipses

$$(\underline{x}' - \mu_i' R)\ R'S_i^{-1}R\ (\underline{x} - R'\mu_i) = 1 \qquad (5.4.3)$$

where R is as defined in ELLPSE. The technique employed to calculate these points is similar to the one employed before. Like ELLPSE, RELPSE also checks for singularity. If it is present, no displays of ellipses appear. It is to be noted that, if the user chooses one of the vectors suggested to him in the display (the vectors identified in the CLUSTR program), the ellipses constructed in RELPSE, if the user depresses the corresponding key, is quite flat, as intended. It is conceivable that, in such an instance, singularities could occur (though we did not see any in our examples) and hence the program checks for them.

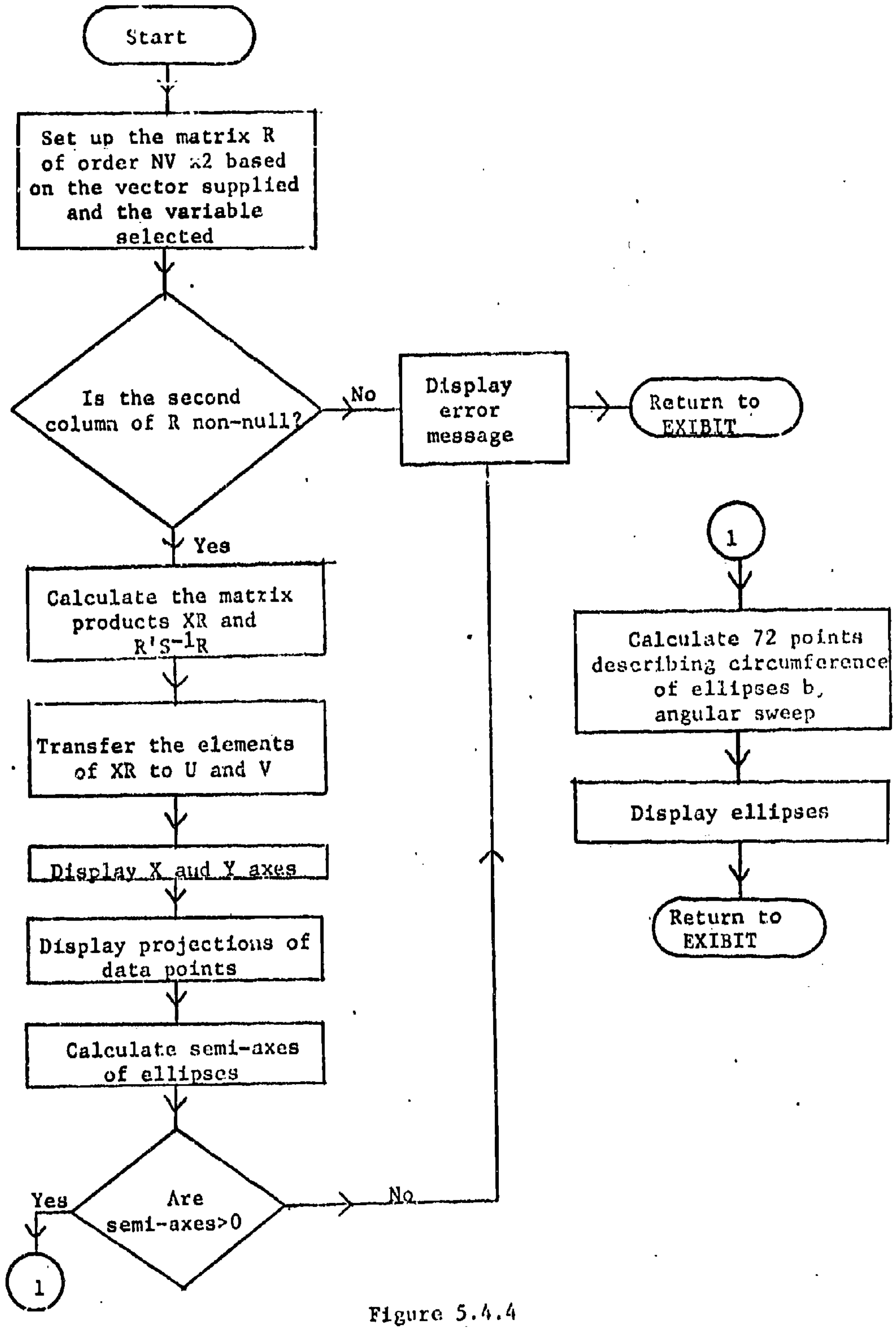
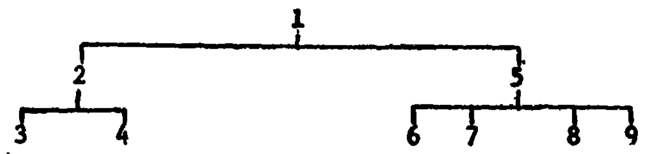

Start
Set up the matrix R of order NV x2 based on the vector supplied and the variable selected
Is the second column of R non-null?
No
Display error message
Return to EXIBIT
Yes
Calculate the matrix products XR and R'S-1R
Transfer the elements of XR to U and V
Display X and Y axes
Display projections of data points
Calculate semi-axes of ellipses
Are semi-axes>0
Yes
No
1
1
Calculate 72 points describing circumference of ellipses by angular sweep
Display ellipses
Return to EXIBIT

Figure 5.4.4

Overlay Structure

To reduce storage requirements, an overlay structure was designed as follows:



1. (Root) contains the main program and Function KD.
2. Contains subroutine CALC
3. Contains subroutine COR1S
4. Contains subroutine COR2
5. Contains subroutine EXIBIT
6. Contains subroutine MESSGE
7. Contains subroutine KEYIN
8. Contains subroutine RELPSE
9. Contains subroutine ELLPSE

Segment 1, along with the main program and function KD, also contains the system support routines IBCOM#, ARIH#, FIOCS#, ADCON#, and system utilities IHCUATBL, IHCUOPT and IHCTRCH. Segment 5 contains all of the GRAF routines required except BUFRS, CUR$$, RCUR$, READSC, SCNDVDK and SCTDV, which are included in segment 7. With the help of this overlay structure and equivalencing of a few arrays in the ELLPSE subroutine, it was possible to reduce the storage requirements to 64K bytes.

Deck Layout for ELLIPSE

```
//STEP1 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(ROOT),SPACE=(TRK,(150,10,5)),            C
               UNIT=SYSDA,DISP=(NEW,PASS)

//FORT.SYSIN DD *
```

```
Main program here
/*
//STEP5 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA3),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Function KD Source Deck
/*
//STEP2 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA),DISP=(MOD.PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine CALC Source Deck
/*
//STEP7 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA5),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine COR1S Source Deck
/*
//STEP6 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA4),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine COR2 Source Deck
/*
//STEP4 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA2),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine EXIBIT Source Deck
Subroutine EXIBIT Source Deck
/*
//STEPO EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&(CHAIN(LINKA8),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine MESSGE Source Deck
/*
//STEP10 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA9),disp=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine KEYIN Source Deck
/*
//STEP3 EXEC FORTGC
//FORT.SYSLIN DD DSNAME==&&CHAIN(LINKA1),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine ELLPSE Source Deck
/*
//STEP8 EXEC FORTGC
//FORT.SYSLIN DD DSNAME=&&CHAIN(LINKA6),DISP=(MOD,PASS),UNIT=SYSDA
//FORT.SYSIN DD *
Subroutine RELPSE Source Deck
/*
//S10 EXEC LKED,PARM=(LET,LIST,OVLY,XREF)
//LKED.SYSLMOD DD DSN=SYS1.GRAPHLIB(ELLIPSE),DISP=SHR,                C
//              SPACE=(TRK,(0,0)
//LKED.SYSLIB DD DSN=SYS1.GRAFLIB,DISP=SHR
```

```
// DD DSN=SYS1.UGALIB,DISP=SHR
// DD DSN=SYS1.SSPLIB,DISP=SHR
// DD DSN=SYS1.FORTLIB,DISP=SHR
// DD DSN=SYS1.LINKLIB,DISP=SHR
// DD DSN=SYS1.GRAPHLIB,DISP=SHR
//LKED.MODULE DD DSN=&&CHAIN,DISP=OLD
//LKED.SYSIN DD *
 INCLUDE MODULE(ROOT)
 INCLUDE MODULE(LINKA3)
 INCLUDE SYSLIB(IBCOM#)
 INCLUDE SYSLIB(ARITH#)
 INCLUDE SYSLIB(FIOCS#)
 INCLUDE SYSLIB(ADCON#)
 INCLUDE SYSLIB(IHCUATBL)
 INCLUDE SYLIB(IHCUOPT)
 INCLUDE SYSLIB(ERRMON)
 INCLUDE SYSLIB(IHCTRCH)
 OVERLAY ONE
 INCLUDE MODULE(LINKA)
 INCLUDE SYSLIB(SINV)
 INCLUDE SYSLIB(MFSO)
 OVERLAY TWO
 INCLUDE MODULE (LINKA5)
 OVERLAY TWO
 INCLUDE MODULE (LINKA4)
 OVERLAY ONE
 INCLUDE MODULE (LINKA2)
INCLUDE SYSLIB(GAFERR)
INCLUDE SYSLIB(LIGHTS)
INCLUDE SYSLIB(WRFMT$)
INCLUDE SYSLIB(DETEKT)
INCLUDE SYSLIB(PLOT)
INCLUDE SYSLIB(DETAIN)
INCLUDE SYSLIB(DISPLA)
INCLUDE SYSLIB($$OVER)
INCLUDE SYSLIB(CHAR)
INCLUDE SYSLIB(POINT)
INCLUDE SYSLIB(LINE)
INCLUDE SYSLIB(PLACE)
INCLUDE SYSLIB($$$$BT)
INCLUDE SYSLIB($$INIT)
INCLUDE SYSLIB(DUMMY$)
INCLUDE SYSLIB($VOVER)
INCLUDE SYSLIB(CLOSE)
INCLUDE SYSLIB(LINE$$)
INCLUDE SYSLIB(UNPLOT)
INCLUDE SYSLIB(PLACE$)
INCLUDE SYSLIB(POINT$)
INCLUDE SYSLIB(ERASE)
INCLUDE SYSLIB(BLANK)
INCLUDE SYSLIB(RESET)
OVERLAY TWOA
INCLUDE MODULE (LINKA8)
OVERLAY TWOA
```

```
INCLUDE MODULE (LINKA9)
INCLUDE SYSLIB(SCNDVDK)
INCLUDE SYSLIB(CUR$$)
INCLUDE SYSLIB(BUFRS)
INCLUDE SYSLIB(SCTDV)
INCLUDE SYSLIB(RCUR$)
INCLUDE SYSLIB(READSC)
OVERLAY TWOA
INCLUDE MODULE (LINKA6)
OVERLAY TWOA
INCLUDE MODULE (LINKA1)
```

# CHAPTER VI

# USER'S GUIDE

## 6.1 Introduction

The user who is interested in using the programs described in the previous chapter would find himself in one of the following situations:

(i) He may not have analysed his multivariate data and may not yet have identified clusters and subspaces. If so, he should first subject his data to the CLUSTR program. The next section contains instructions on how to use (execute) this program.

(ii) He may have analysed his data using the CLUSTR program but has not loaded the data set for the ELLIPSE program. If so, he should execute the IEBGENER Utility routine and load the data set. This is described in section 6.3.

(iii) Finally, the user may have gone through the steps (i) and (ii) above and may want to see the displays of projected clusters and subspaces. The use of the ELLIPSE program including an indication of what to look for in the displays is described in section 6.4.

Section 6.5 contains an illustration.

## 6.2 The CLUSTR program

The analysis of the user's multivariate data begins with the identification of clusters and overdetermined subspaces, if any. For this purpose, the user must first analyse his data using the CLUSTR program.

This is a batch program. The following data cards need to be supplied:

Data Card 1

Columns 1-3, number of points (individuals or experimental units)

Columns 6-7, number of variables (responses) measured on each experimental unit

Columns 8-11, alpha level for cluster core on first pass (suggested value 0.90)

Columns 13-16, alpha level for cluster extension on first pass (suggested value 0.50)

Columns 18-21, alpha level for cluster core on second pass (suggested value 0.90)

Columns 23-26, alpha level for cluster extension on second pass (suggested value 0.50)

Columns 28-31, alpha level for cluster core on third pass (suggested value 0.90)

Columns 33-36, alpha level for cluster extension on third pass (suggested value 0.50)

Data Card 2

This is a variable format card and should contain the FORMAT by which each experimental unit is to be read.

The remaining data cards contain the observations, one card (or record which may consist of several cards) contains the coordinates of one point. The numbering of these points is implicit, according to the sequential order of these cards.

Our suggestion above that 0.90 should be used as alpha level for cluster core and 0.50 as alpha level for cluster extension is empirical. Of course, he can use any other set of values. For a detailed discussion of this matter the reader is directed to Graney [10].

The output of this program has been discussed at length in section 5.2 The punched deck produced by this program is required for the ELLIPSE program.

## 6.3 IEBGENER Utility routine

As explained in section 5.3, it is necessary to execute the IEBGENER Utility routine to load the output of the CLUSTR program into a data set required as input to the ELLIPSE program. The deck set-up for the execution of this utility routine is as follows:

```
(i)     JOB card
(ii)    //STEPG EXEC PGM=IEBGENER
(iii)   //SYSPRINT DD SYSOUT=A
(iv)    //SYSIN DD DUMMY
(v)     //SYSUT2 DD DSN=SYS1.R2250,VOL=SER=UGA231,DISP=SHR,UNIT=2314
(vi)    //SYSUT1 DD DATA,DCB=(RECFM=FB,LRECL=80,BLKSIZE=320)
(vii)   Data Cards
(viii)  /*
```

The data cards consist of a header card followed by the punched deck produced by the CLUSTR program (of course, the user could produce his own data cards, and any assignment of points to clusters or subspaces which he desires. In this respect the CLUSTR program is merely intended to give him guidance--but a very strong one indeed). The header card is made up as follows (all numbers right justified);

Columns 1-4, numbers of points (individuals or experimental units)

Columns 5-8, number of variables (responses)measured on each experimental unit

Columns 9-12, number of clusters identified by the CLUSTR program

Columns 13-16, number of overdetermined subspaces (simple structure solutions) identified by CLUSTR program

After the IEBGENER Utility routine is executed, one is ready for the ELLIPSE program.

## 6.4 The ELLIPSE program

This program works under the control of GMS (Graphics Monitor System). For greater detail on the operation of GMS see Penn [31]. In order to be operational under the conversational GMS, the load module of the program has to be a member of the partitioned data set GRAPHLIB. The user should verify, by typing the command $NAMES on the console typewriter, that the program, in fact, is a member of the GRAPHLIB data set. If not, the user will first have to compile and link edit the program using the deck set-up given in section 5.4. The user can then execute the program using the command $LINK ELLIPSE to link to it.

Photographs made during the use of the program are reproduced here. The user will find it helpful to refer to them while studying the rest of the section. Some of the photographs will be specifically discussed in the next section.

The execution of ELLIPSE begins with the display of an informative message. The user should carefully read the message. (Note especially the use of the programmed function key 30. This is to be pressed only when it is desired to stop the execution of the program.) The user should then press any key other than 0 or 30. He is now asked to type the coordinates of a vector which he wishes to utilize in order to form a

```
THIS PROGRAM DISPLAYS CLUSTERS BY PROJECTING THEM ON
VARIOUS 2-DIMENSIONAL SUBSPACES.SIMPLE STRUCTURE
SOLUTIONS CAN ALSO BE INDICATED.REFER TO YOUR
INSTRUCTION CHART.HAVE YOU ENTERED ALL NECESSARY
DATA VIA IEBGENER?
          THE BOTTOM ROW OF THE PROGRAM
FUNCTION KEYS IS LIT UP.THEY WILL BE USED
AS DIRECTED.THE DARK ONE,KEY NO. 30,IS
THE PANIC BUTTON.IT WILL RETURN YOU TO THE
MONITOR.

IN CASE OF PANIC PRESS KEY 30
NOW PRESS ANY KEY TO GET STARTED
```

```
   ACCORDING TO THE ROTATION PROGRAM THE
FOLLOWING VECTORS TRANSFORM RAW DATA INTO
SIMPLE STRUCTURE SOLUTIONS NO.1 TO  3
 -0.309-0.275 0.856-0.309
  0.012 0.843-0.155 0.516
 -0.306-0.259-0.316 0.860

ENTER A TRANSFORMATION VECTOR.
NO MORE THAN 7 CHARACTERS,DECIMAL POINT
MUST BE TYPED
DEPRESS JUMP KEY AFTER EACH COORDINATE
X(1)=_
X(2)=
X(3)=
X(4)=
X(5)=
X(6)=
X(7)=
X(8)=
IF PREVIOUS VECTOR OK,JUST PRESS EOB
```

```
   ACCORDING TO THE ROTATION PROGRAM THE
FOLLOWING VECTORS TRANSFORM RAW DATA INTO
SIMPLE STRUCTURE SOLUTIONS NO.1 TO  3
 -0.309-0.275 0.856-0.309
  0.012 0.843-0.155 0.516
 -0.306-0.259-0.316 0.860

ENTER A TRANSFORMATION VECTOR.
NO MORE THAN 7 CHARACTERS,DECIMAL POINT
MUST BE TYPED
DEPRESS JUMP KEY AFTER EACH COORDINATE
X(1)=-0.306
X(2)=-0.259
X(3)=-0.316
X(4)=0.860_
X(5)=
X(6)=
X(7)=
X(8)=
IF PREVIOUS VECTOR OK,JUST PRESS EOB
```

```
 PRESS ONE KEY CORRESPONDING TO THE AXIS
- VARIABLE YOU WISH TO SELECT, OR 30 IF
 YOU WISH TO STOP.IF YOU WISH TO MAKE
CHANGES IN YOUR VECTOR PRESS KEY 28.
```

2-dimensional plane, the other axis being an observable variable. Notice that vectors which transform the original data points into overdetermined subspaces appear in this display for the user's ready reference. The user should try one or more of these vectors but he may also supply any number of other vectors, if he feels that this would help him interpret the structure of his data. The coordinates of the vector intended to be used should be typed in through the alphameric keyboard. The place where the digit (or any character for that matter) typed will appear on the screen is indicated by a cursor. The use of the "JUMP" key after one coordinate is entered, will cause the cursor to move over to the place for the next coordinate. After all the coordinates of a vector are entered, the user should press EOB. This is done by pressing both the 'ALT' and the '5' key of the alphameric keyboard. The first time the user is asked to supply a vector, he must give a non-null vector. (Later on, null vectors will be acceptable and simply mean that there is no change. At that time the user would just press EOB when this display appears and thus indicate that he does not wish to change the previous vector.)

After the vector is supplied, the user is asked to choose a variable as the other axis of a 2-dimensional plane. The choice is made by pressing the programmed function key corresponding to the number of the variable; if variable number 1 is desired, press key 1, etc.

After a vector and a variable have thus been chosen, the desired 2-dimensional plane will appear on the screen. The abscissa corresponds to the variable, the ordinate corresponds to the chosen vector. The legend (numbers given alongside the coordinate axes) indicates minimum and maximum values. The serial number of each data point is projected at the appropriate coordinates. Ellipses, i.e., projections of unit ellipsoids, based upon the within-cluster metric of all points, are drawn around each cluster center. These ellipses generalize the concept of a standard unit interval in one dimension.

One aspect to be observed on the screen at this time is whether the points which supposedly belong to the same cluster, are close together (regardless which axes or vectors are used), and whether one could distinguish them visually from points belonging to a different cluster. There may be some overlaps but the clusters should be visually distinct. If certain points exhibit the above phenomenon in all displays, then they can be regarded as forming a cluster.

The user has now a choice. He may wish to superimpose one of the overdetermined subspaces, or he may wish to change the vector, or the variable, or both. To change the vector, he presses key 29, to change the number of the variable, he presses key 31, and to superimpose an overdetermined subspace, he presses the key corresponding to the number of that subspace.

Superimposition of an overdetermined subspace results in the display of projections of unit ellipsoids having metric based on only those points which belong to the overdetermined subspace. If the plane of projection selected is orthogonal to the overdetermined subspace, the projections of unit ellipsoids under reference will be flat and elongated. Further, the projections of the points lying on the subspace will make a narrow band (almost resembling a straight line). The plane of projection would be orthogonal to the overdetermined subspace, if the vector which transforms the original data points into this overdetermined subspace is supplied as the desired vector. As many different planes as there are dimensions can be constructed by taking this normal vector against each of the variable axes.

The plane of projection, in a way, is the direction from which we look at the ellipsoids embedded in the p-dimensional space. The ellipsoids having metric based on the points belonging to an overdetermined subspace must necessarily be flat and elongated. However, they must be viewed from the proper direction. Otherwise, this elongation may not appear or, in some instances, they will appear as very small circles.

Once again, the user can press key 29 to supply a new vector, key 31 to change the number of the variable and the key corresponding to the number of an overdetermined subspace to superimpose the subspace. The program continues in this manner. It can be terminated at any time by pressing key 30.

It should be noted that the projections onto a plane formed by any pair of observable variables is a special case. If the user desires to have projections onto the plane formed by variables 1 and 2, say, all he has to do is supply (1.0, 0.0, . . . , 0.0) as the vector and press key 2. In fact, since the program does not necessarily require normalized vectors as input, any vector of the form (a, 0, 0, . . . , 0) where $a \neq 0$, results in the selection of variable 1 as one of the axes.

## 6.5 An Illustration

The programs described above were used in the analysis of artificial data. The data consisted of 45 points and 4 variables. These data were generated in the following manner. First, 180 normal deviates with zero mean and unit variance were generated; they made up the 180 elements of a 45 x 4 matrix, numbered column-wise. The first 25 measurements on the 4th variable were then redefined by the relation

$$z(I) = z(I)/10 + (z(I - 135) + z(I - 90) + z(I - 45))/3$$

$$I = 136, 137, \ldots, 160$$

i.e., the first 25 observations on variable 4 were replaced by one tenth of the original observation plus the mean of the first three variables. Similarly, the last 25 observations on variable 3 were replaced by the relation

$$z(I) = z(I)/10 + (z(I - 90) + z(I - 45) + z(I - 45))/3$$

$$I = 111, 112, \ldots, 135$$

The modification of the data matrix by these two operations built in 2 subspaces. In effect, the first 25 observations on variable 4 became almost a linear combination of the remaining 3 variables and the last 25 observations on variable 3 similarly became almost a linear combination of the remaining 3 variables. Still, however, all the variables had zero mean, and to introduce mean shifts, the vectors (3, 9, 5, 9), (5, 9, 7, 3) and (7, 3, 7, 5) were added to the first 15 observations, second 15 observations and the last 15 observations respectively. Thus the entire data matrix became a simulated sample drawn from normal populations with mean vectors (3, 9, 5, 9), (5, 9, 7, 3) and (7, 3, 7, 5) and two built in subspaces. This data matrix was then subjected to the first program of cluster identification and subspace determination. This program correctly assigned the first 15 points to one cluster, the second 15 points to another cluster and the last 15 points to a third cluster. The subspace identification part of the program gave 3 overdetermined subspaces. This was not at all surprising since the two planes were already built in and, as frequently happens in such instances (see section 4.2), a plane was found somewhat in-between the two constructed planes. The cosines between the normals to these three planes were as follows:

| | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1.00000 | -0.29684 | 0.00673 |
| 2 | -0.29684 | 1.00000 | 0.02303 |
| 3 | 0.00673 | 0.02303 | 1.00000 |

The planes identified were as under (serial numbers of points belonging to the planes are given).

Plane 1: --12, 21, 22, 23, 24, 25, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, (24 points)

Plane 2: --2, 9, 14, 15, 17, 18, 29, 30, 32, 34, 37, 39, (12 points)

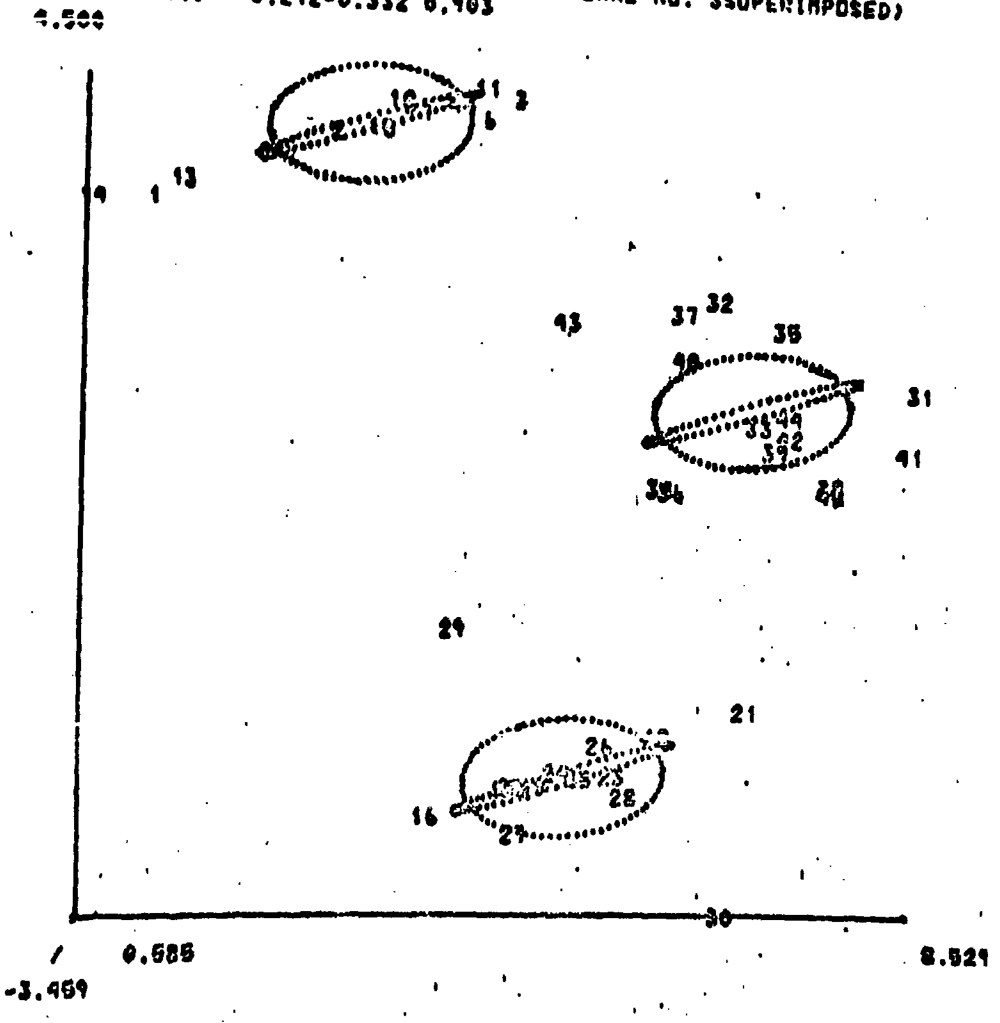
Plane 3: --1, 2, 4, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 24, 25, (20 points)

According to the built-in subspaces, one subspace should have contained the last 25 points, i.e., points 21-45 and the other subspace should have contained the first 25 points. Plane 1 given above did pick up 23 of the last 25 points and an extra point number 12. Likewise, plane 3 picked up 20 of the first 25 points. Plane 2 picked up 6 of the points belonging to plane 1 and 6 of the points belonging to plane 3. Thus plane 2 is somewhat of a mixture of the planes 1 and 3.

The results of the CLUSTR program were then displayed using the ELLIPSE program. Notice especially the following displays in which the 2-dimensional planes were formed by selecting a suggested vector (-0.306, -0.259, -0.316, 0.860) and an observable variable. The vector under consideration is the vector 3, which transformed the original data points into simple structure plane 3.

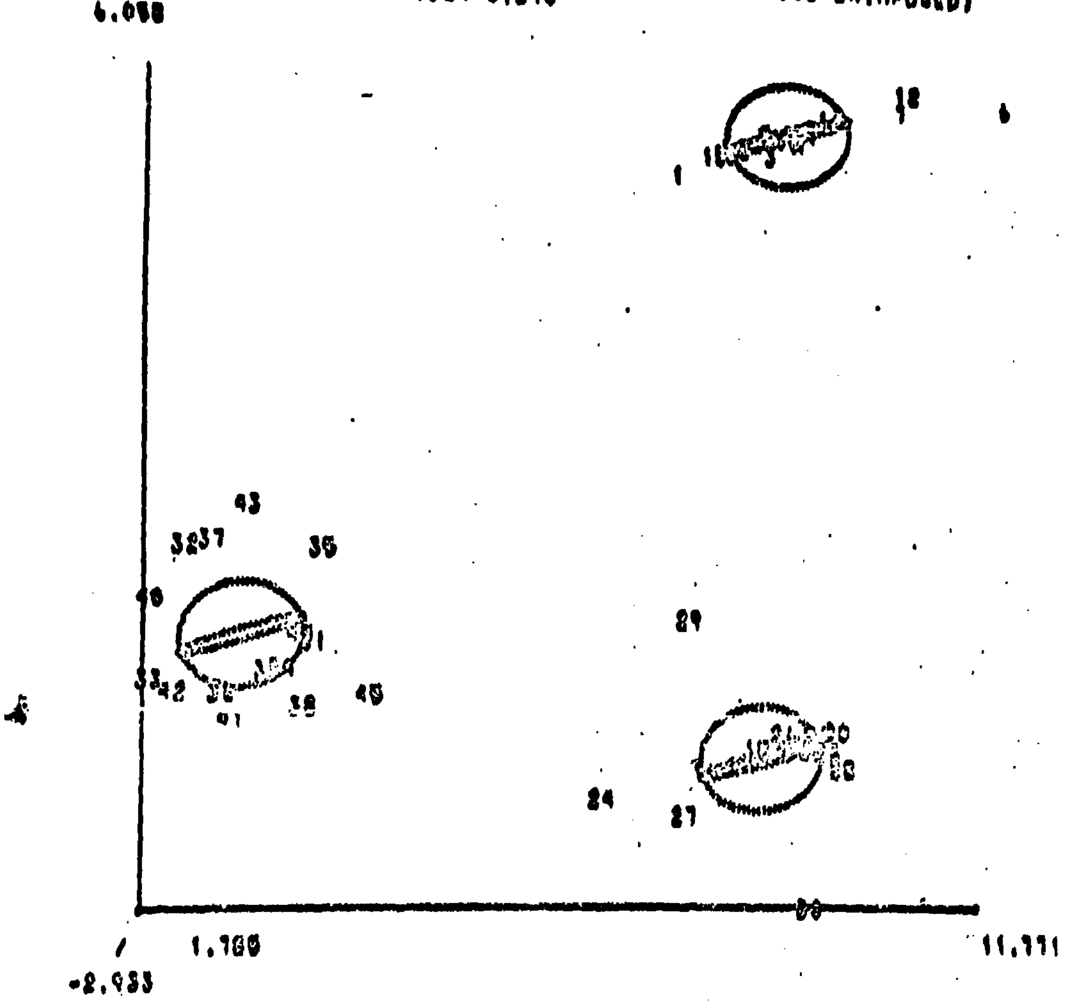
(a) Variable 1 against vector 3--After normalizing, the vector reduced to (0.0, -0.272, -0.332, 0.903). Plane 3 was superimposed. Points number 1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26 can be seen to be lying on the simple structure plane.

VARIABLE 1AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 3SUPERIMPOSED)
0.0 -0.272-0.332 0.903



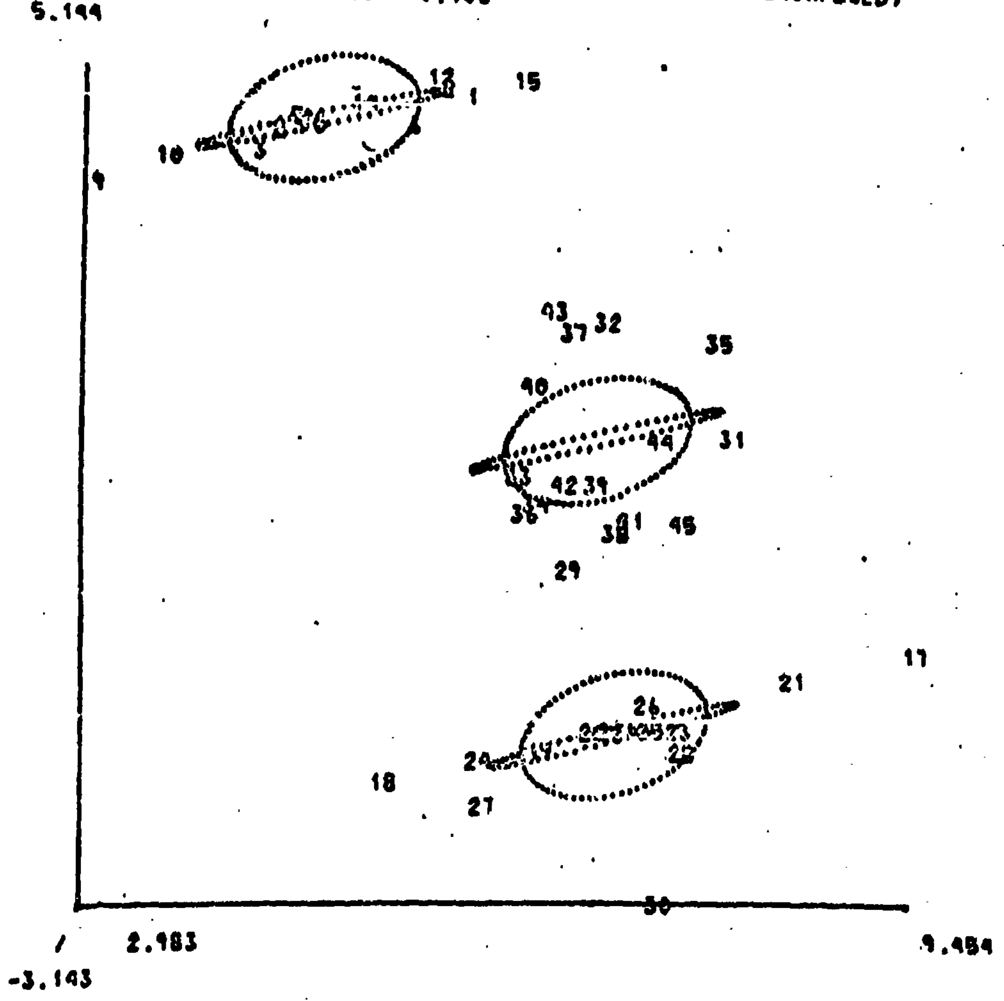
(b) Variable 2 against vector 3--After normalizing, the vector reduced to ( 0.317, 0.0, 0.327, 0.890). Plane 3 was superimposed. Points number 1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26 can be seen lying on the simple structure plane.

```
VARIABLE 2AGAINST VECTOR BELOW:(SIMPLE PLANE NO. 3SUPERIMPOSED)
        -0.311 0.0  -0.327 0.640
```



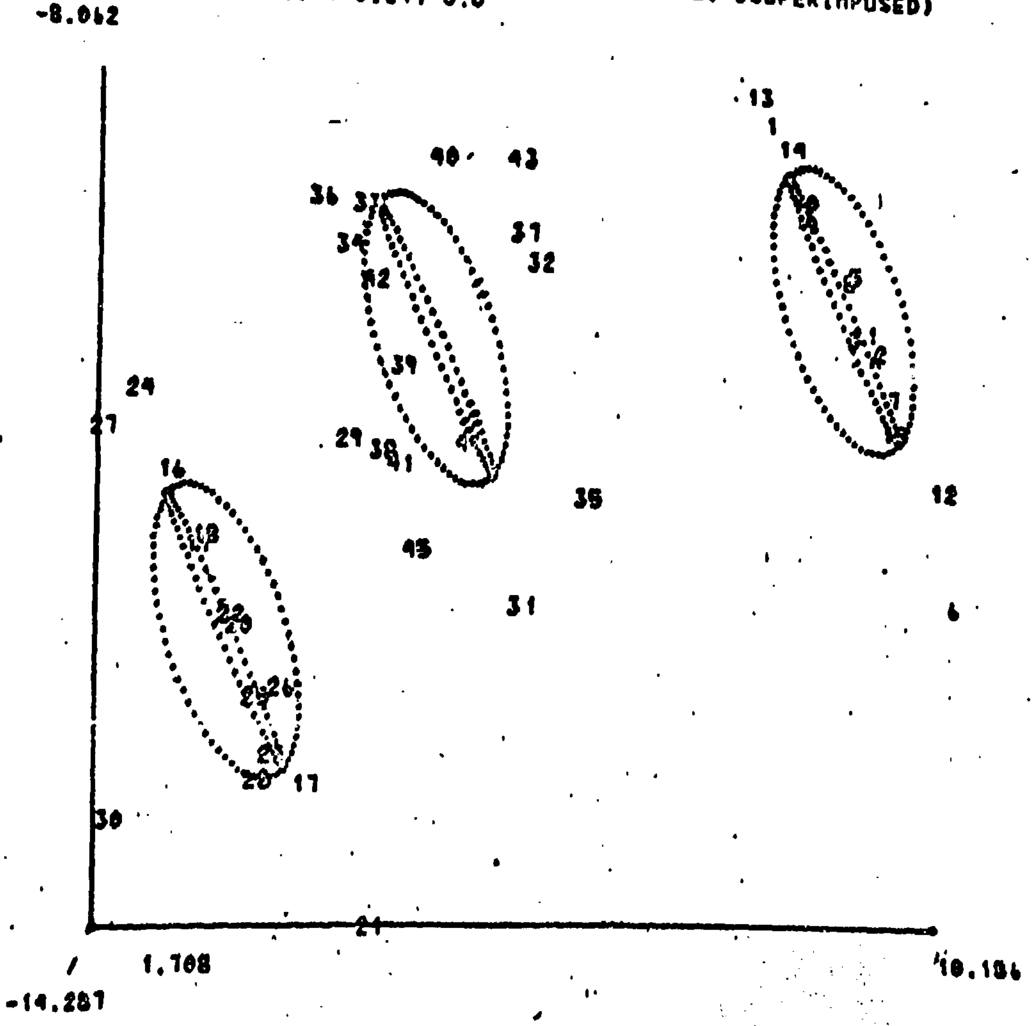
(c) Variable 3 against vector 3--After normalizing, the vector reduced to (-0.322, -0.273, 0.0, 0.906). Plane 3 was superimposed. Points number 1-26 can be seen lying on the simple structure plane.

VARIABLE 3AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 3SUPERIMPOSED)
-0.322-0.273 0.0 0.906

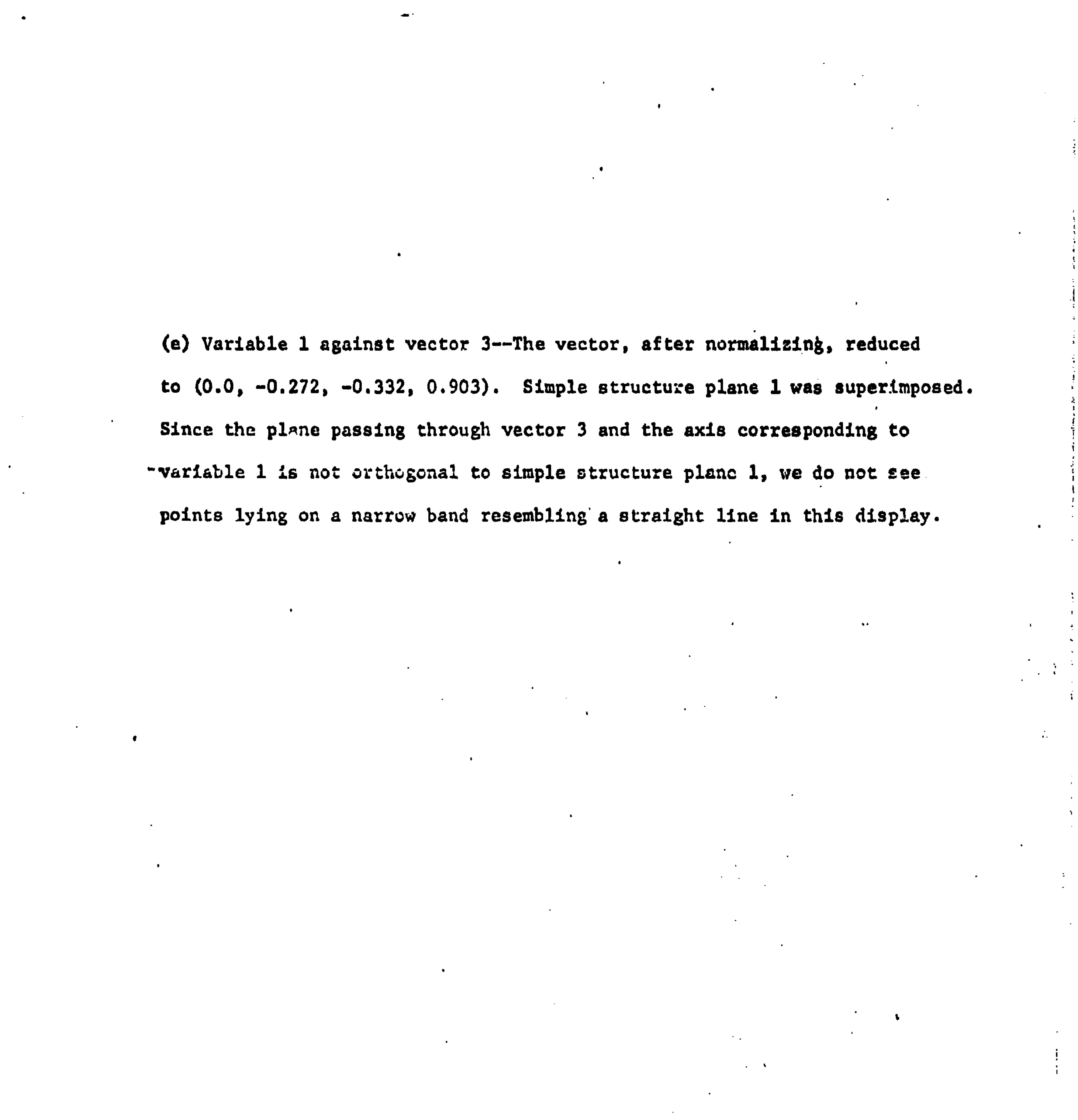


(d) Variable 4 against vector 3--The vector, after normalizing, reduced to (-0.599), -0.507, -0.619, 0.0). Plane 3 was superimposed. Points number 1-27 with the exception of number 5 can be seen lying on the simple structure plane.

VARIABLE 4AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 3SUPERIMPOSED)
-0.549-0.507-0.619 0.0

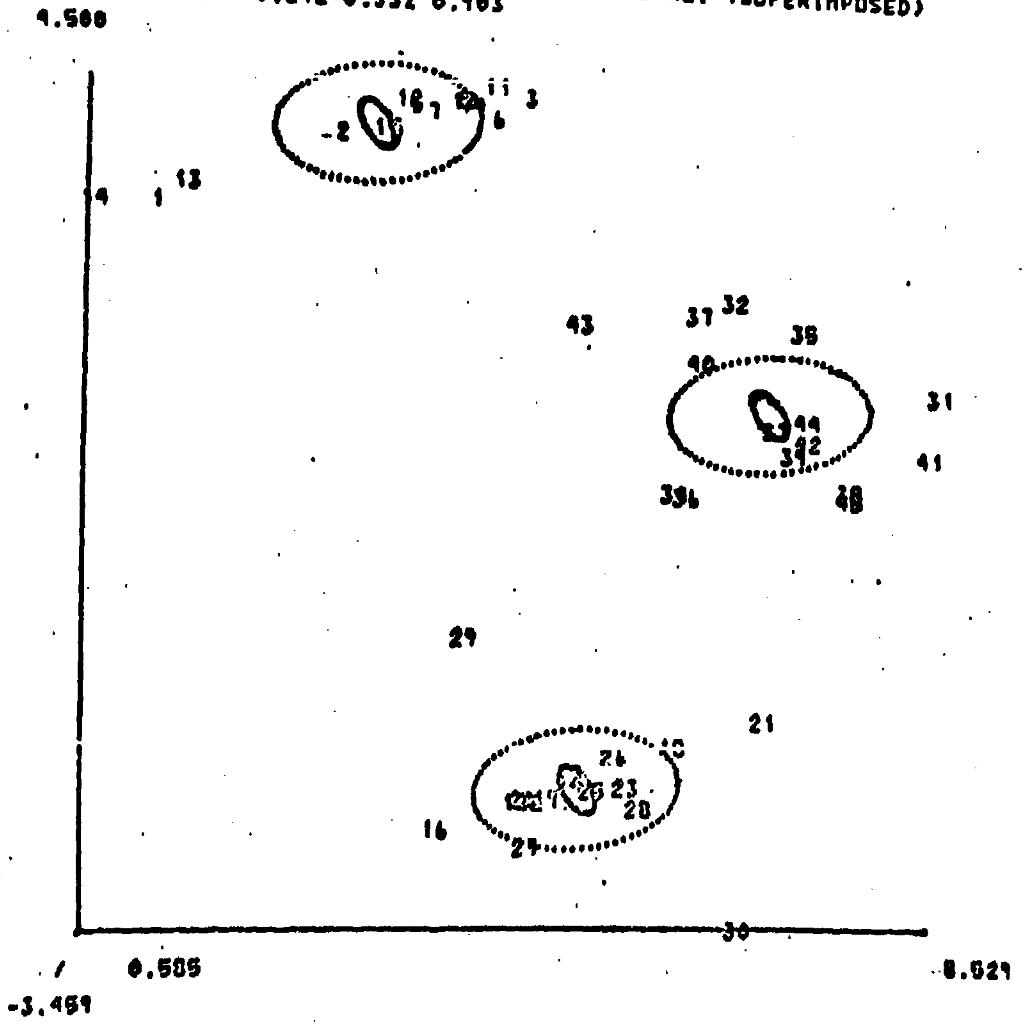


The above 4 displays pertain to each of the 4 variables against vector 3 and superimposition of simple structure plane 3. Vector 3 is the vector which transforms the raw data into simple structure plane 3. Points 1-25 with the exception of 3, 5, 6, 12 and 23 lie on this plane. Vector 3 is the normal to this simple structure plane 3. Hence any plane passing through vector 3 is orthogonal to the simple structure plane 3. When projections onto this orthogonal plane are taken, the points lying on the simple structure plane should fall within a narrow band (almost resembling a straight line) and the above 4 displays clearly bring out this fact. Variables 1, 2, 3, and 4 make 4 different planes, respectively, passing through vector 3 and all orthogonal to the simple structure plane 3. This can also be thought of as rotating a plane passing through vector 3 around the vector 3. Projections are taken when this rotating plane passes through the axis corresponding to each of the variables. Attention should also be drawn to these displays.



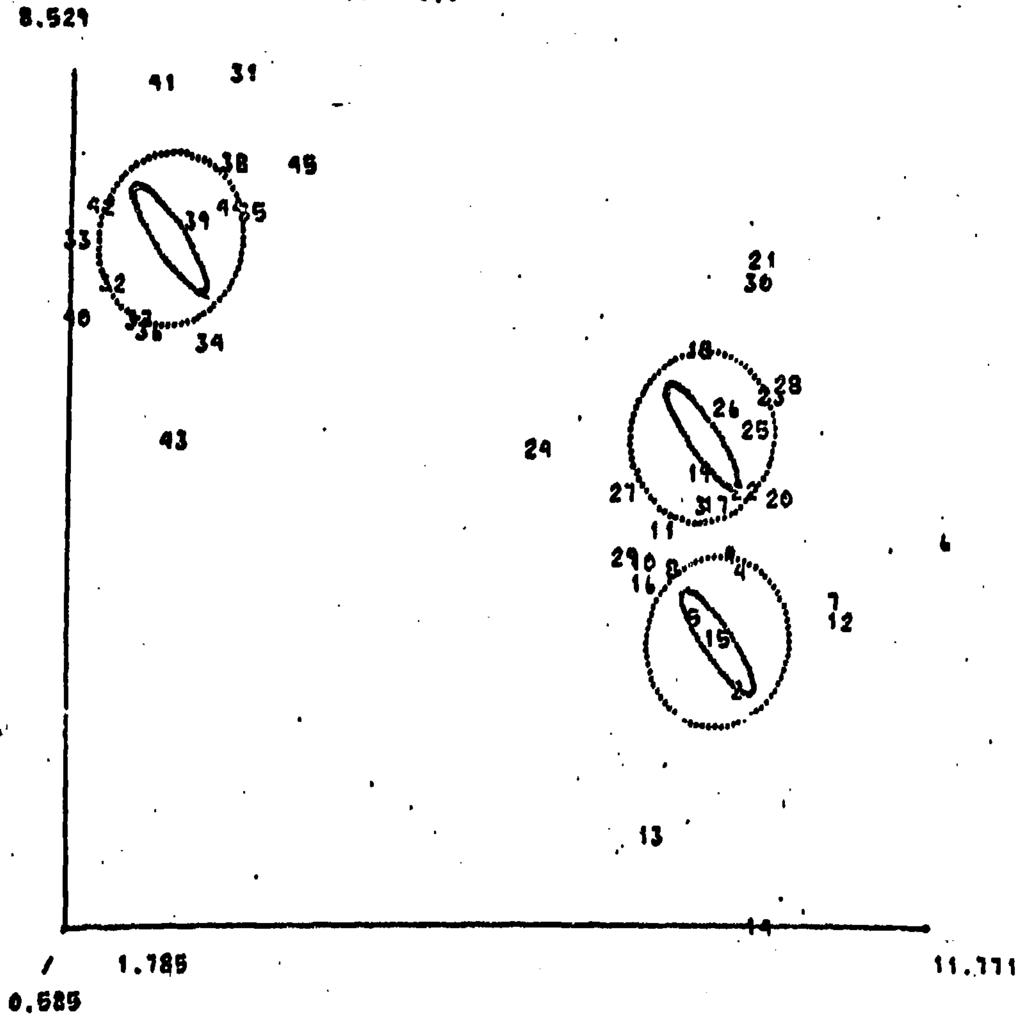
(e) Variable 1 against vector 3--The vector, after normalizing, reduced to (0.0, -0.272, -0.332, 0.903). Simple structure plane 1 was superimposed. Since the plane passing through vector 3 and the axis corresponding to variable 1 is not orthogonal to simple structure plane 1, we do not see points lying on a narrow band resembling a straight line in this display.

VARIABLE 1AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 1SUPERIMPOSED)
0.0 -0.212-0.332 0.903



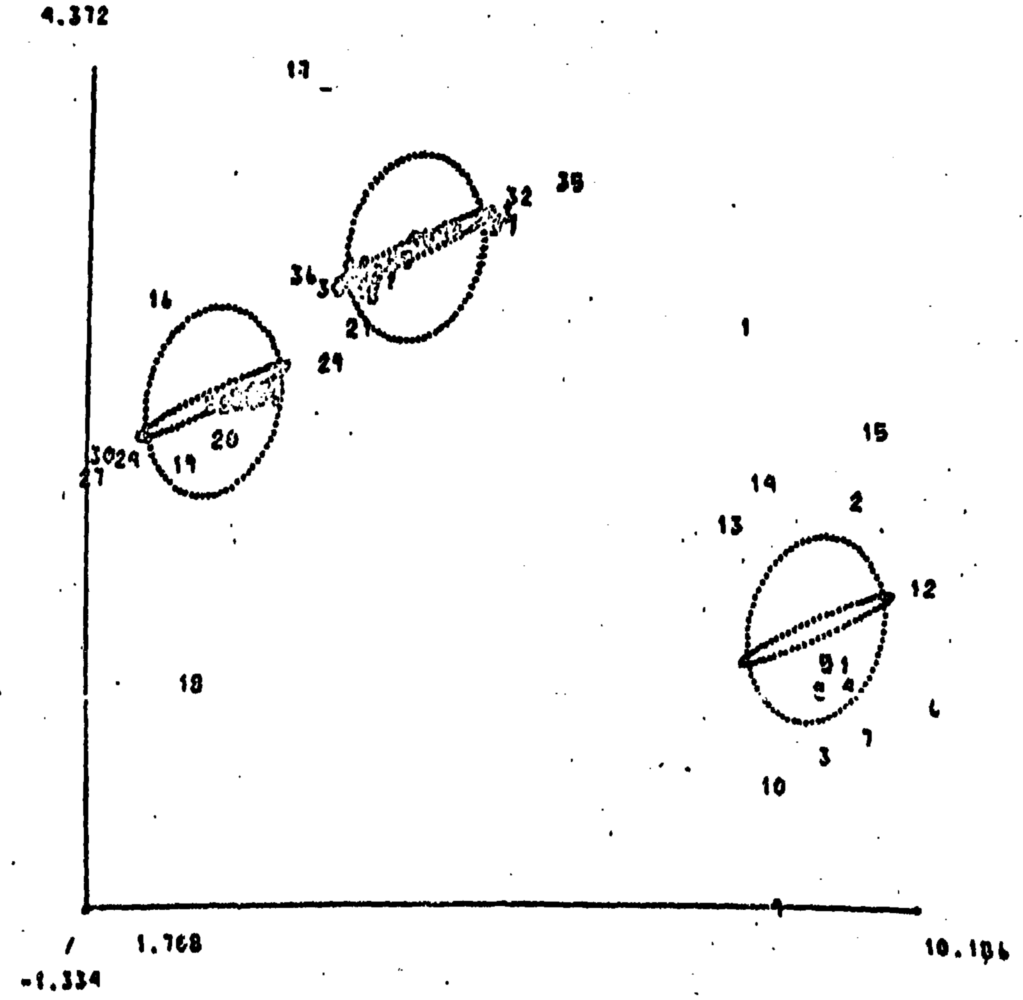
(f) Variable 2 against variable 1--Plane 3 was superimposed. Once again, for the reasons mentioned in (e) above, we do not see a good simple structure in this display. We simply are not looking at the ellipsoids from the proper position.

VARIABLE 2AGAINST VECTOR BELOW:(SIMPLE PLANE NO. 3SUPERIMPOSED)
1.000 0.0 0.0 0.0



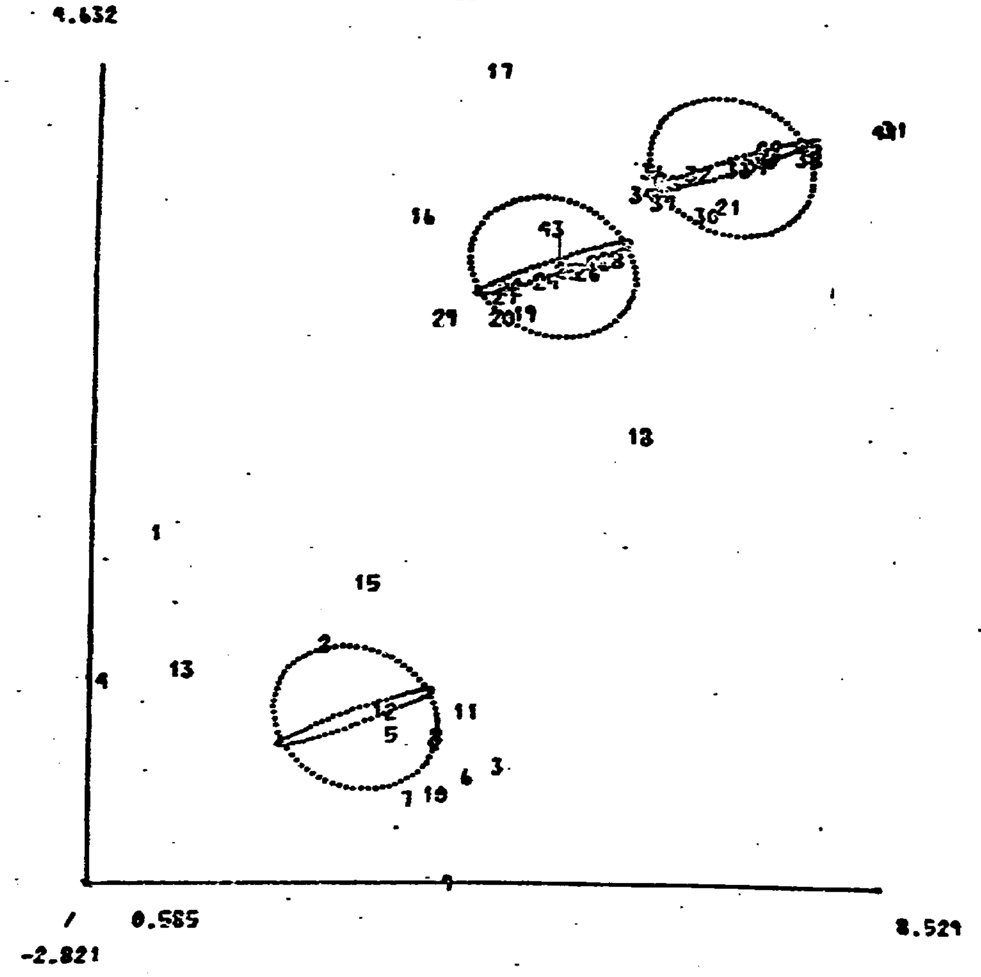
(g) Variable 4 against vector 1--The vector, after normalizing, reduced to (-0.325, -0.289, 0.900, 0.0). Plane 1 was superimposed. Since any plane passing through vector 1 is orthogonal to simple structure plane 1, we expect to see a good simple structure in this display and we do. Points number 21, 22, 23, 25, 26, 28, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, lie within a narrow band resembling a straight line.

VARIABLE 4AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 1SUPERIMPOSED)
-0.325-0.289 0.900 0.0

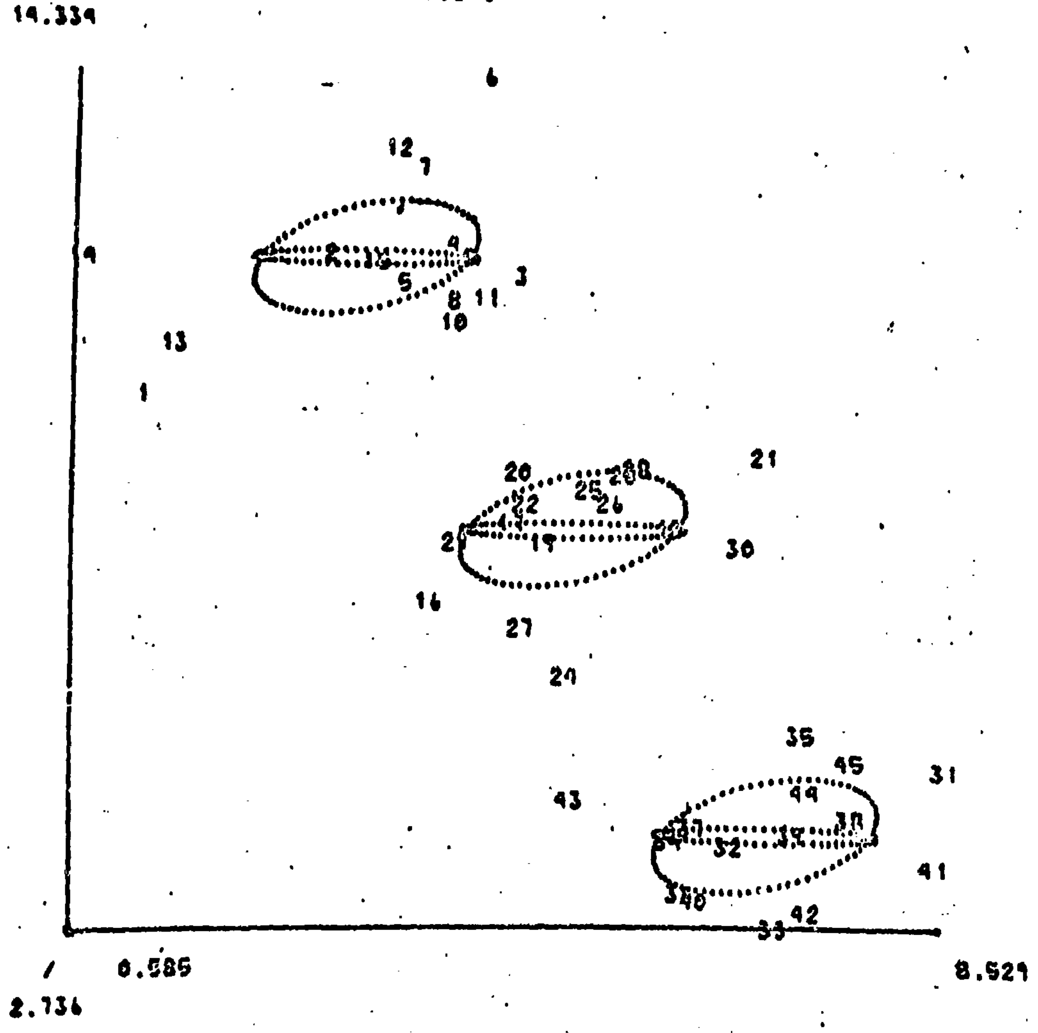


The user should find the other displays easy to understand. A good simple structure is seen only when a plane is selected which passes through a vector that transforms the raw data into a simple structure solution, and the corresponding simple structure plane is superimposed. It should, however, be noted that the clustering of points is not affected by this principle and hence, in all displays, the clusters can be easily identified.

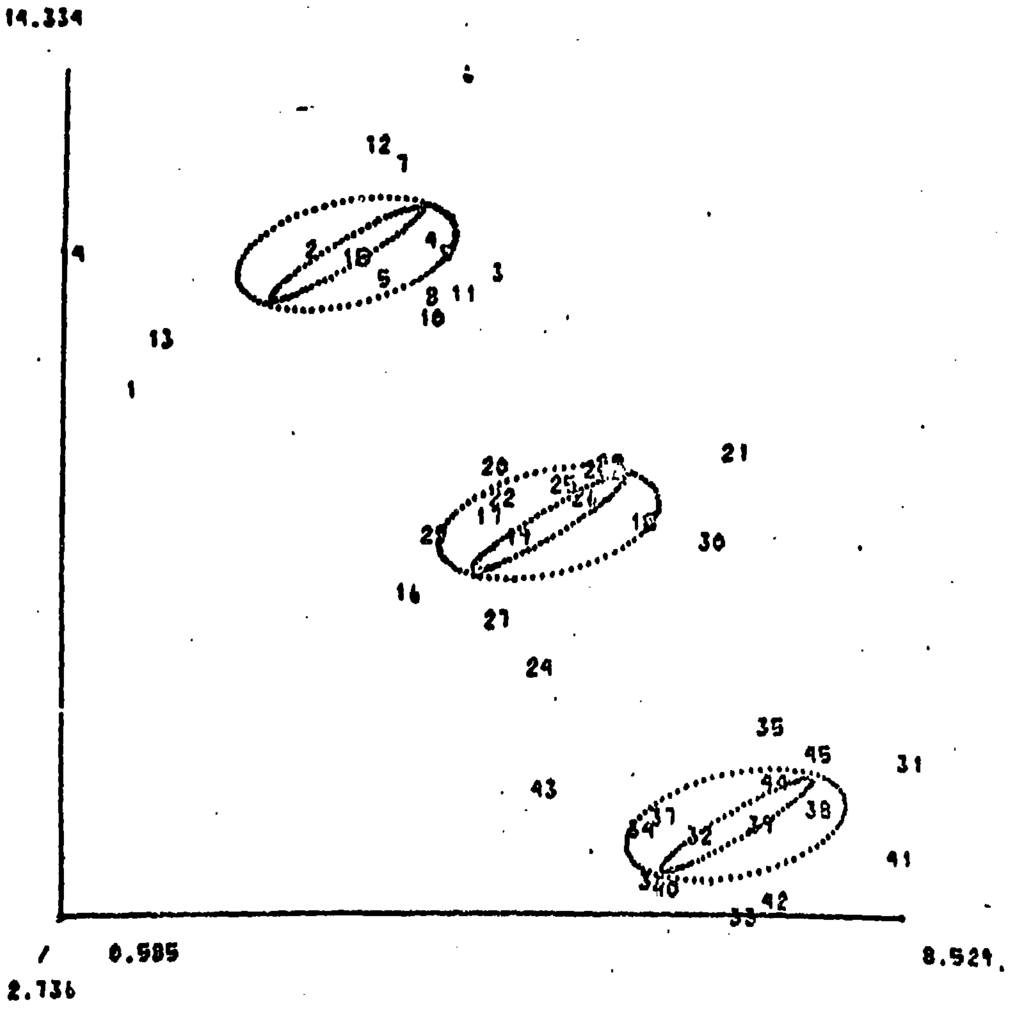
VARIABLE 1AGAINST VECTOR BELOW:(SIMPLE PLANE NO. 1SUPERIMPOSED)
0.6 -0.289 0.900-0.325



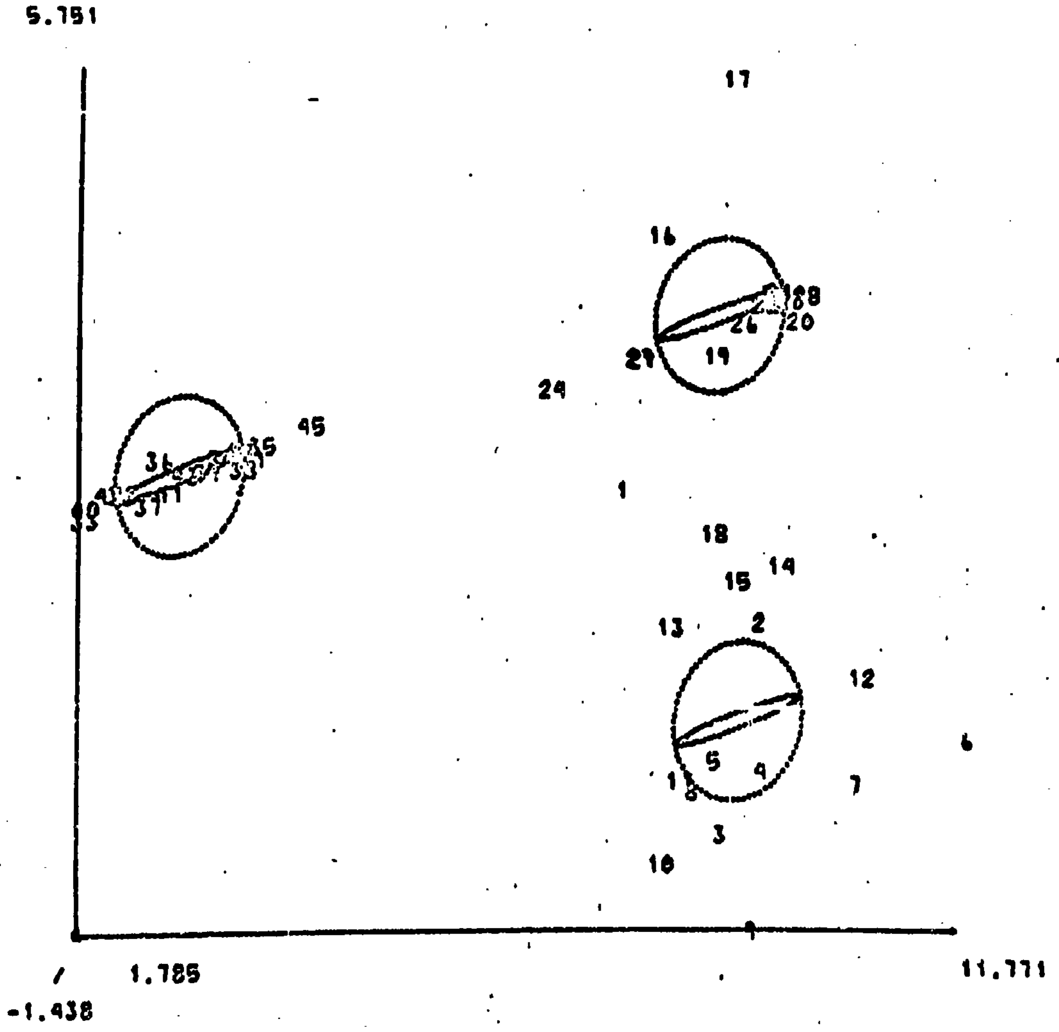
VARIABLE 1AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
0.0 0.893-0.155 0.516



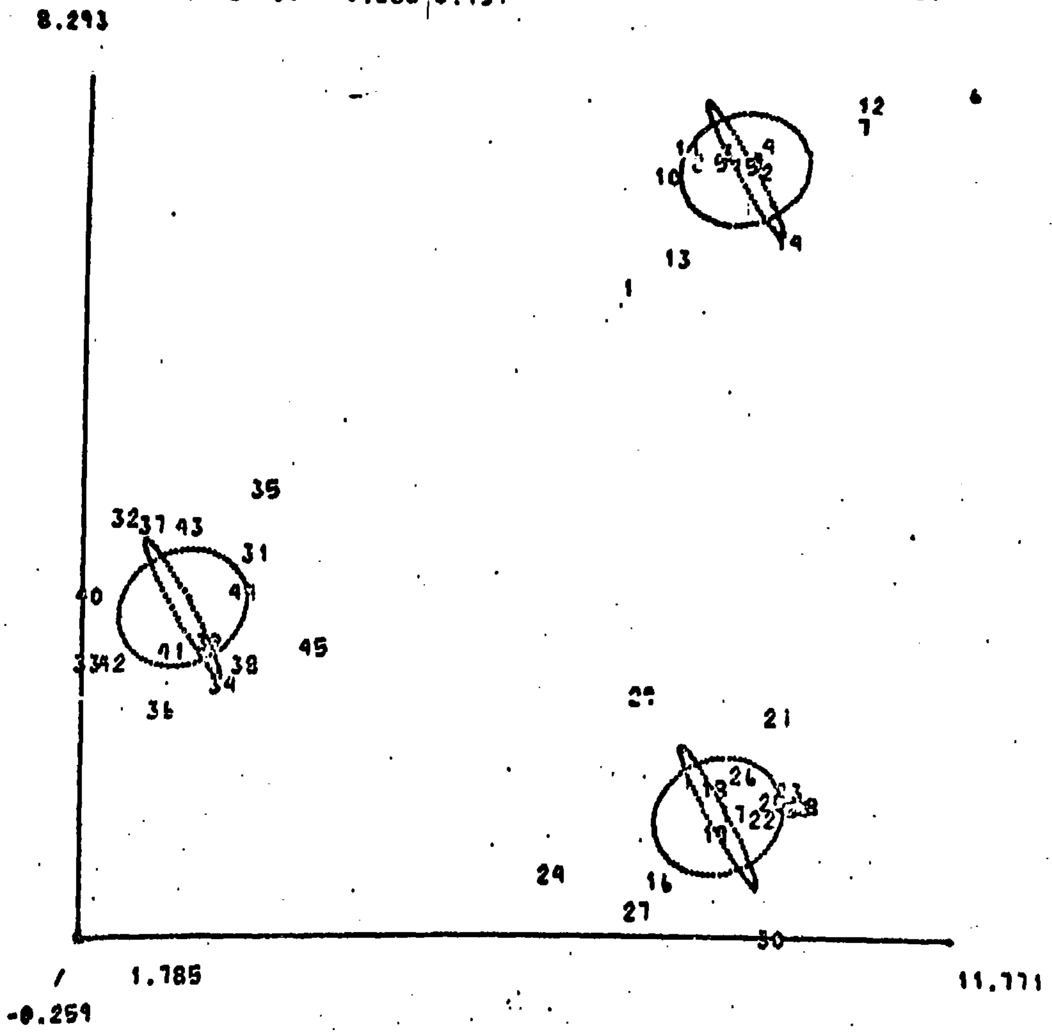
VARIABLE 1AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 3SUPERIMPOSED)
0.0 0.843-0.159 0.516



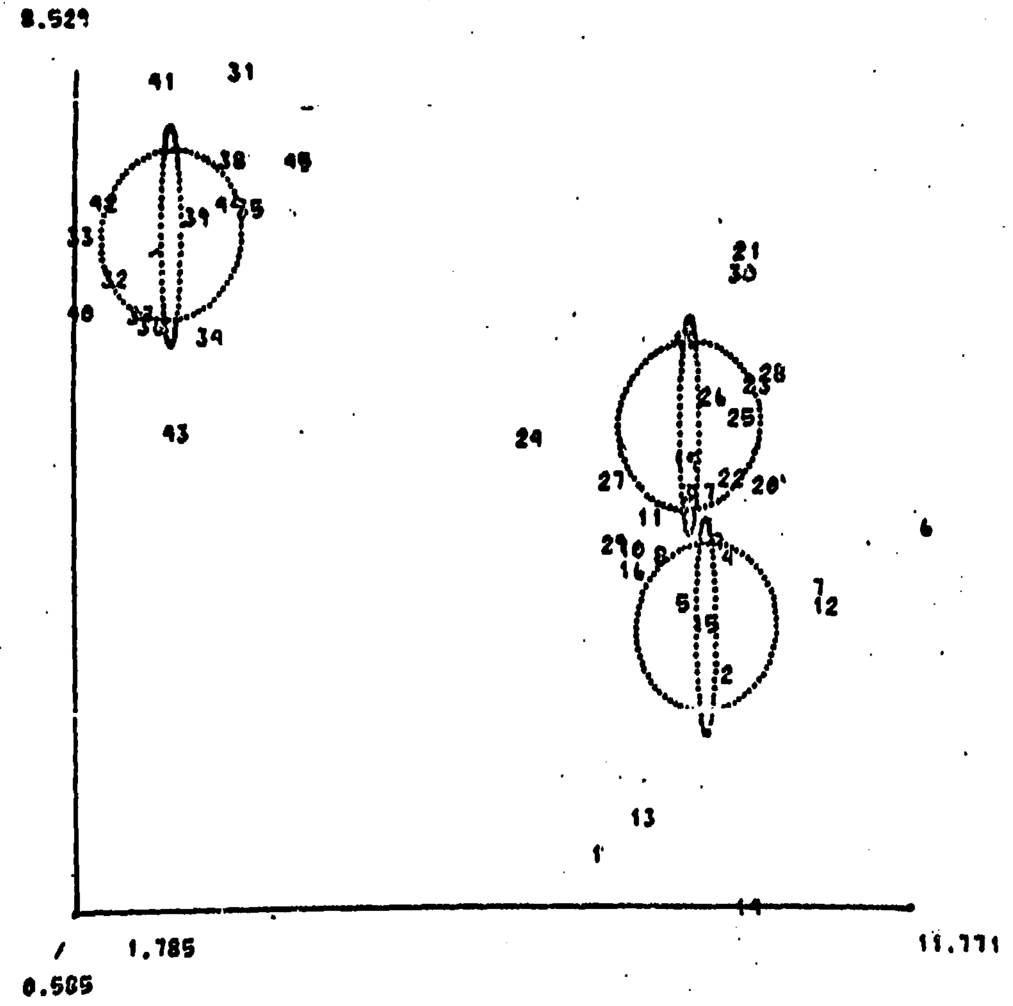
VARIABLE 2AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 1SUPERIMPOSED)
-0.322 0.0 0.891-0.322



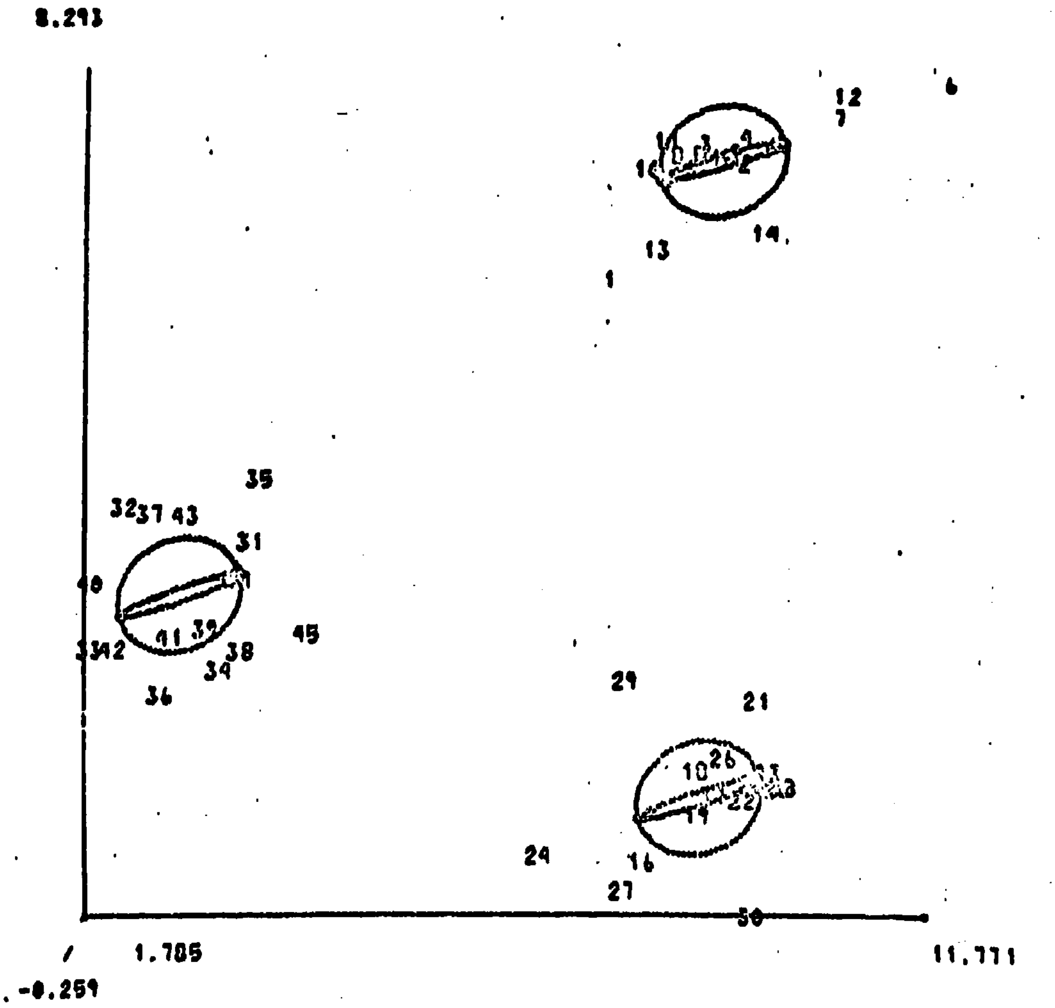
VARIABLE 2AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
0.022 0.0 -0.208 0.951

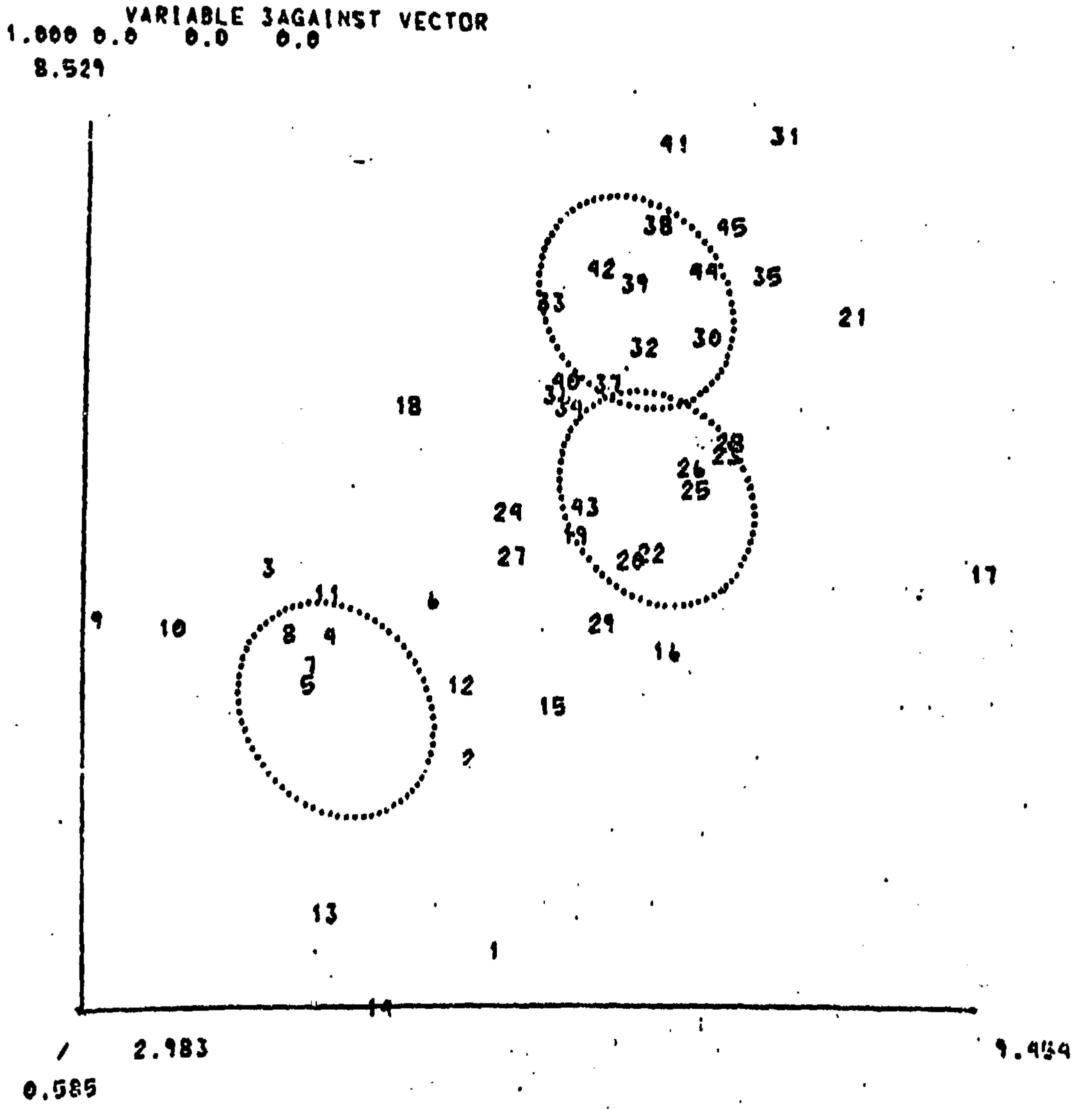


```
VARIABLE 2AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
          1.000 0.0   0.0   0.0
```



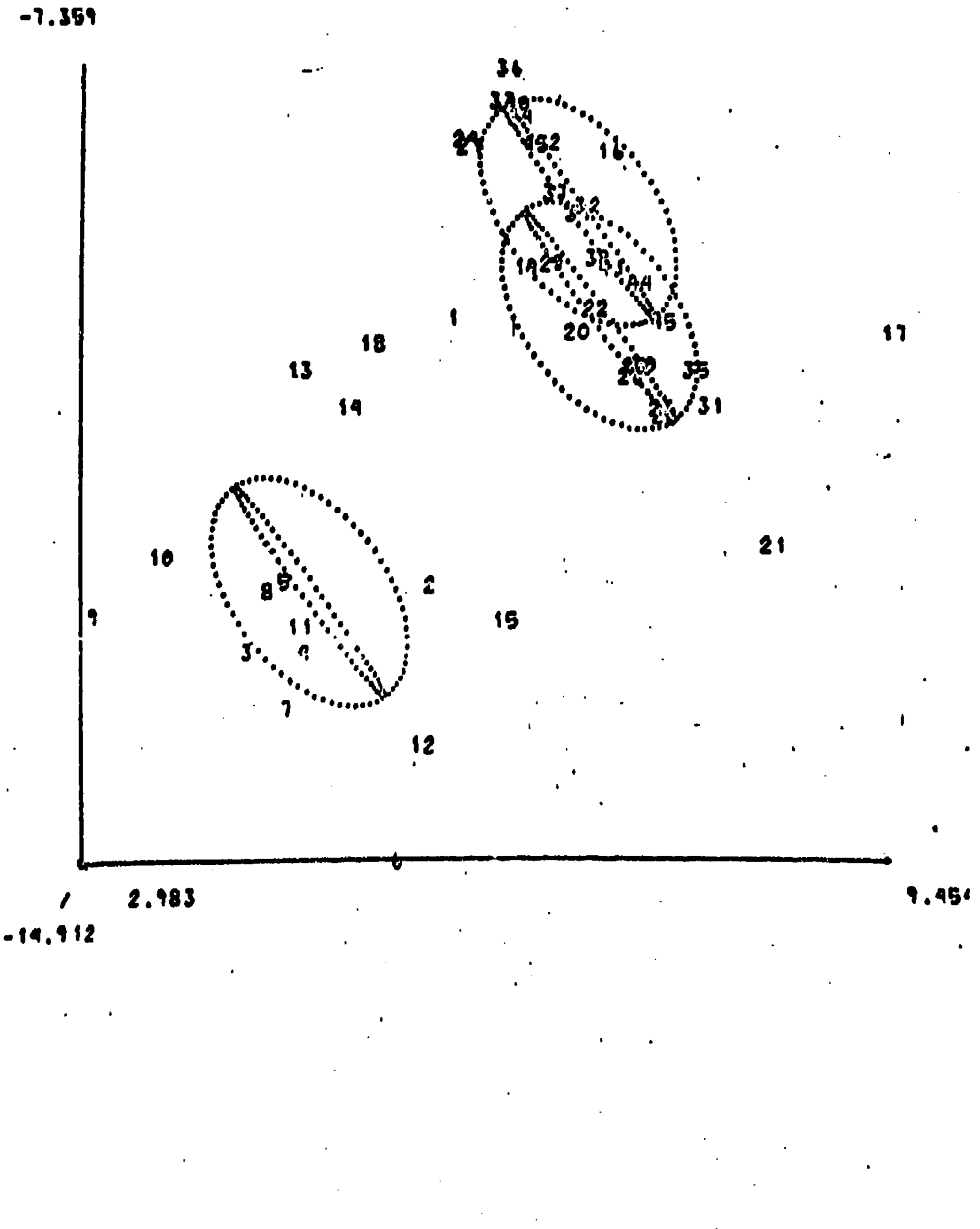
VARIABLE 2AGAINST VECTOR BELOW,(SIMPLE PLANE NO. ;SUPERIMPOSED)
0.022 0.0 -0.280 0.951



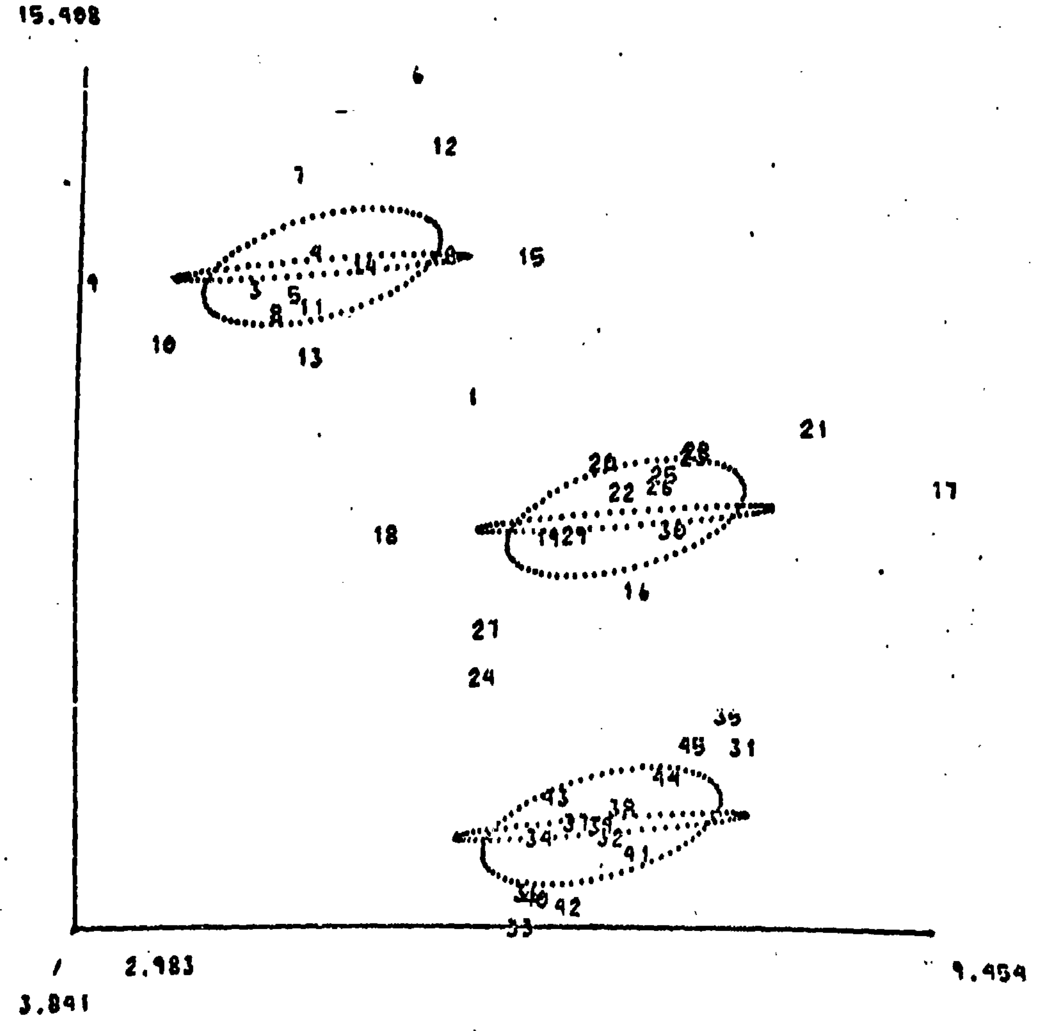


VARIABLE 3AGAINST VECTOR
1.000 0.0 0.0 0.0
8.529
0.585
2.983
9.959
41
31
38
45
42
39
44
35
33
21
32
30
40
37
18
34
28
23
26
25
29
43
19
27
20
22
3
17
11
6
9
10
29
8
4
16
7
5
12
15
2
13
1
14

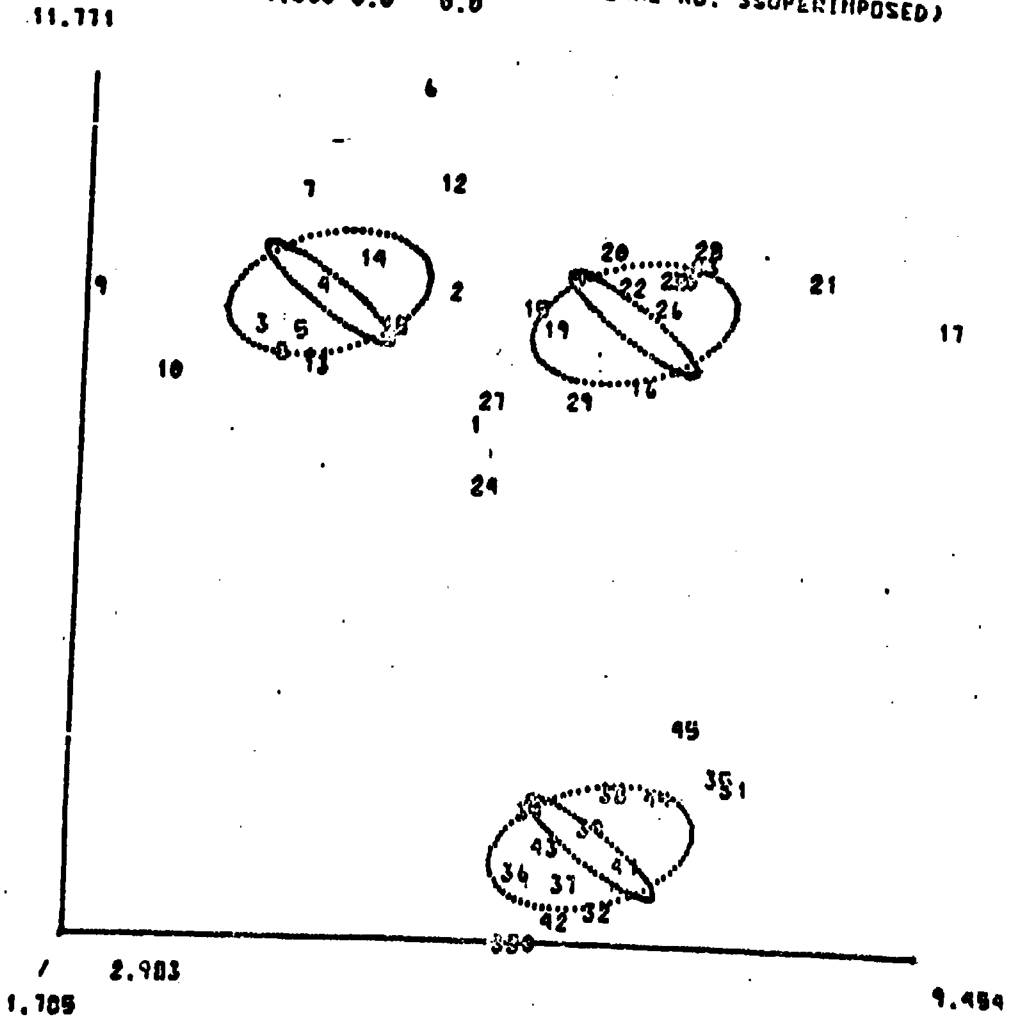
VARIABLE 3AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 1SUPERIMPOSED)
-0.598-0.533 0.0 -0.598



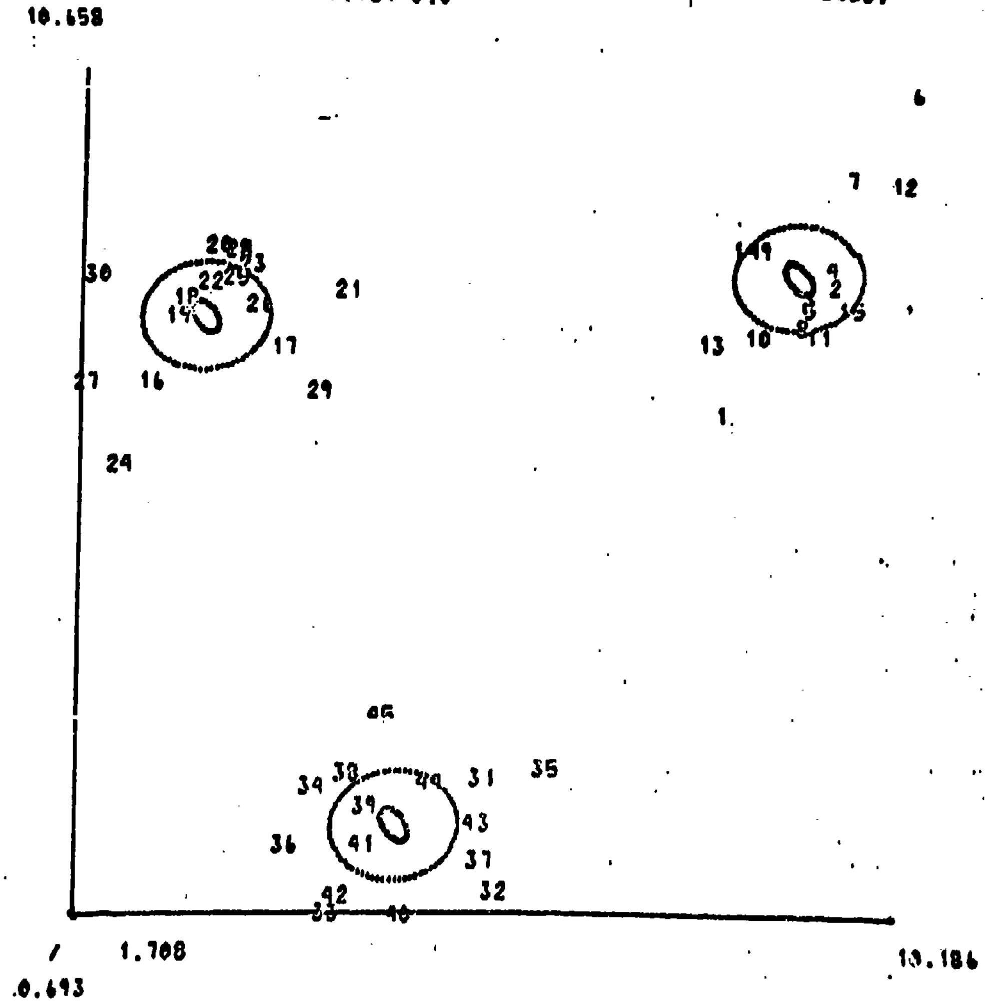
VARIABLE 3AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
0.012 0.853 0.0 0.522



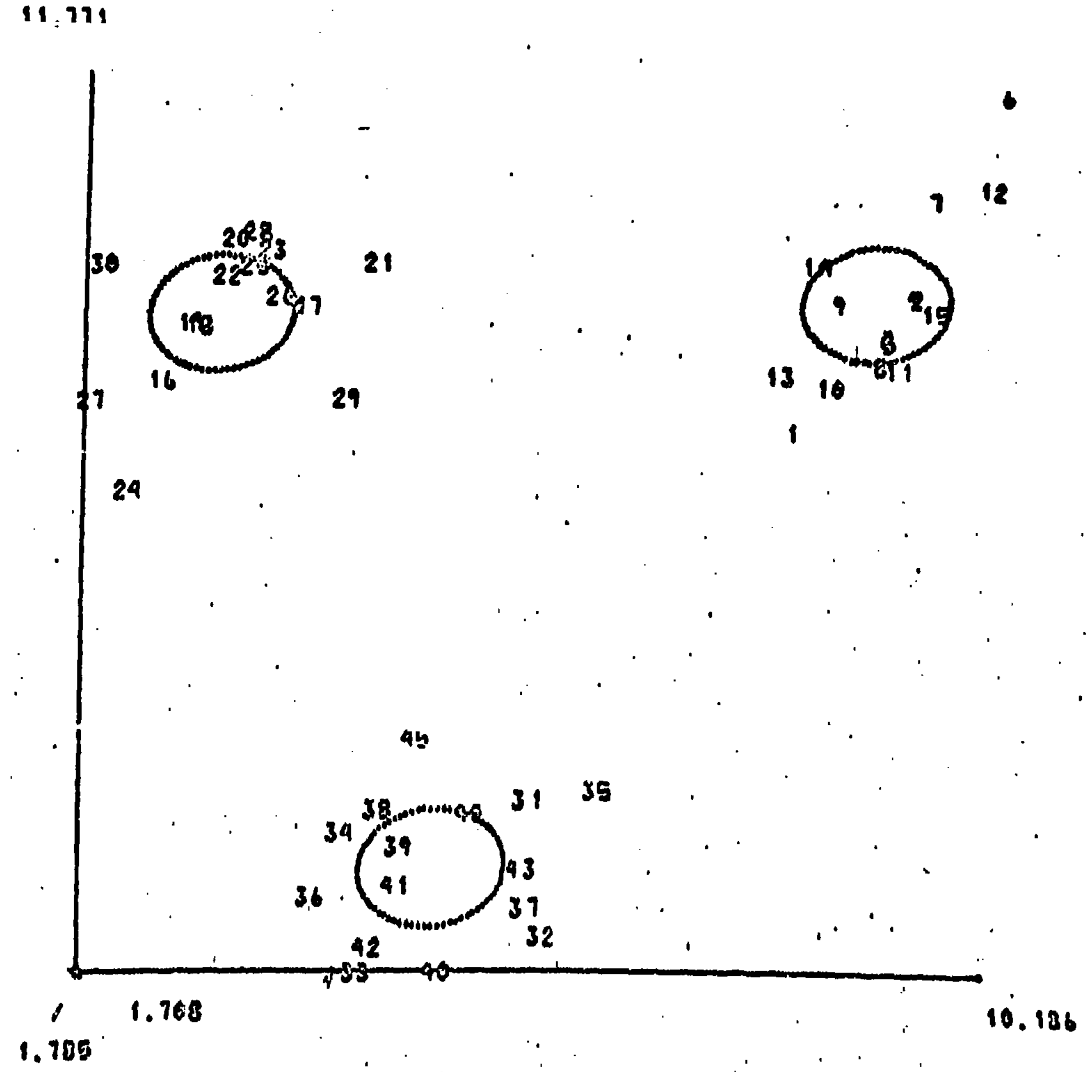
VARIABLE 3AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 3SUPERIMPOSED)
0.0 1.000 0.0 0.0



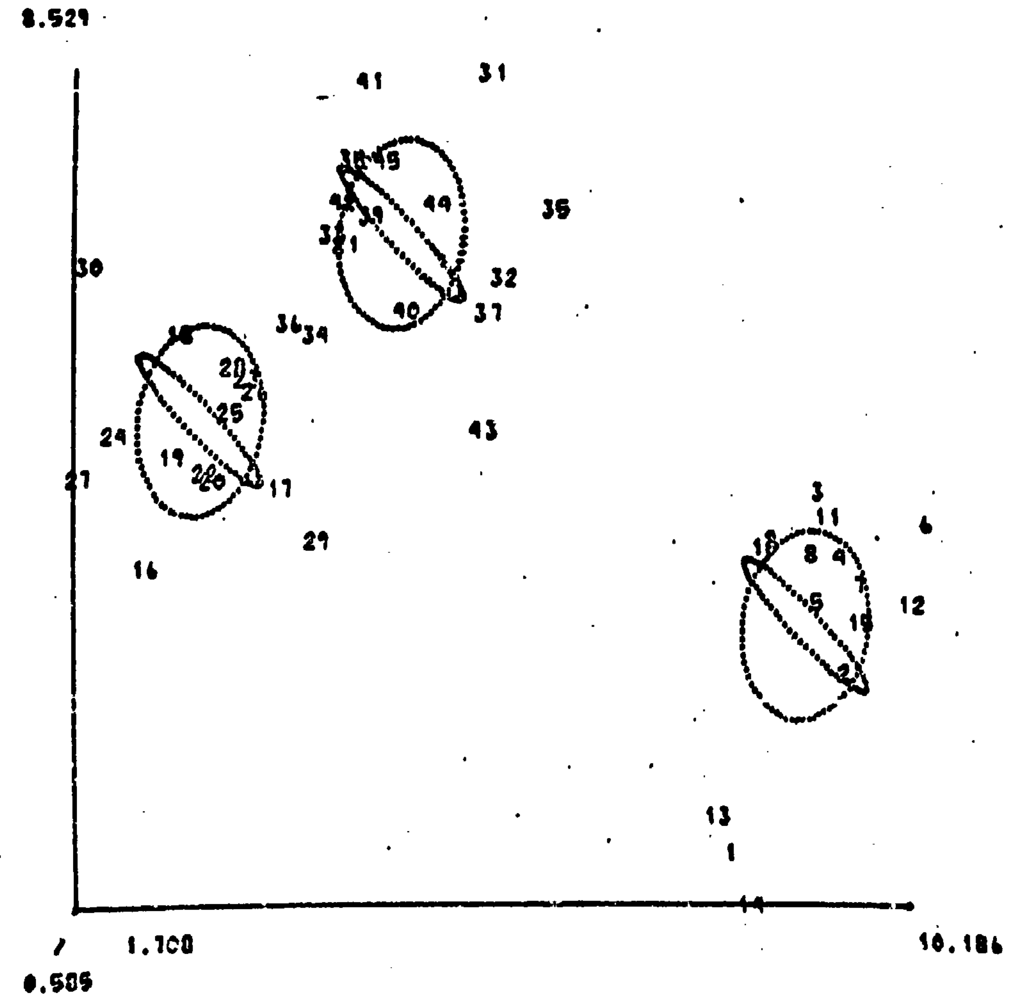
VARIABLE 9AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
0.019 0.983-0.181 0.0



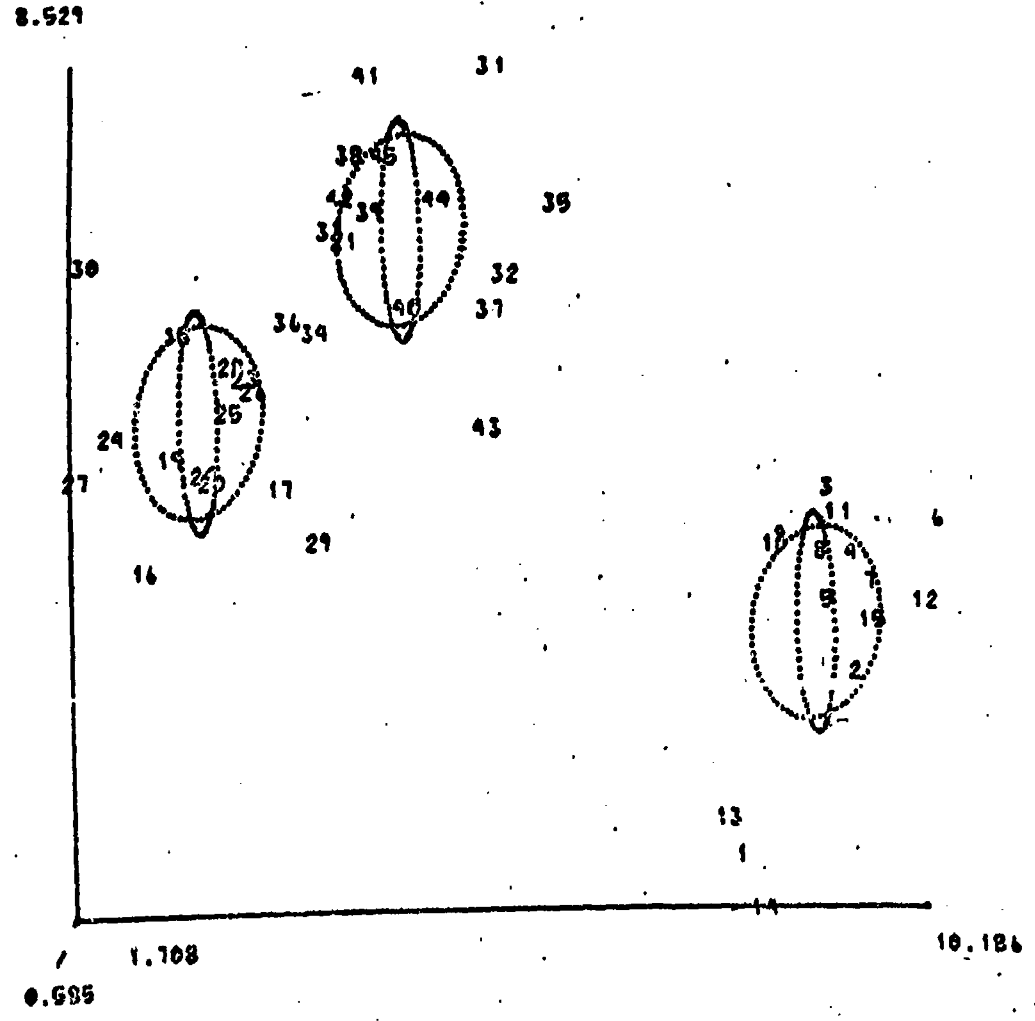
VARIABLE 9AGAINST VECTOR
0.0 1.000 0.0 0.0



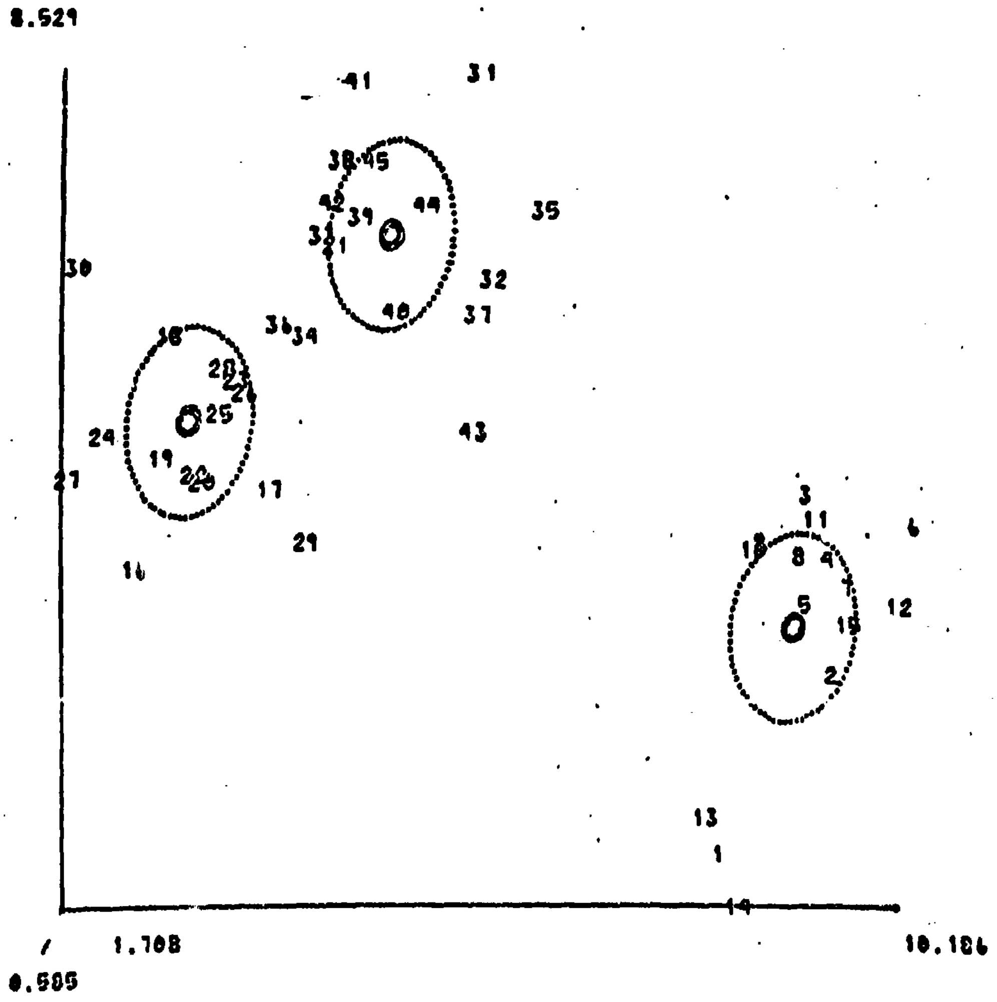
VARIABLE 4AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 1SUPERIMPOSED)
1.000 0.0 0.0 0.0



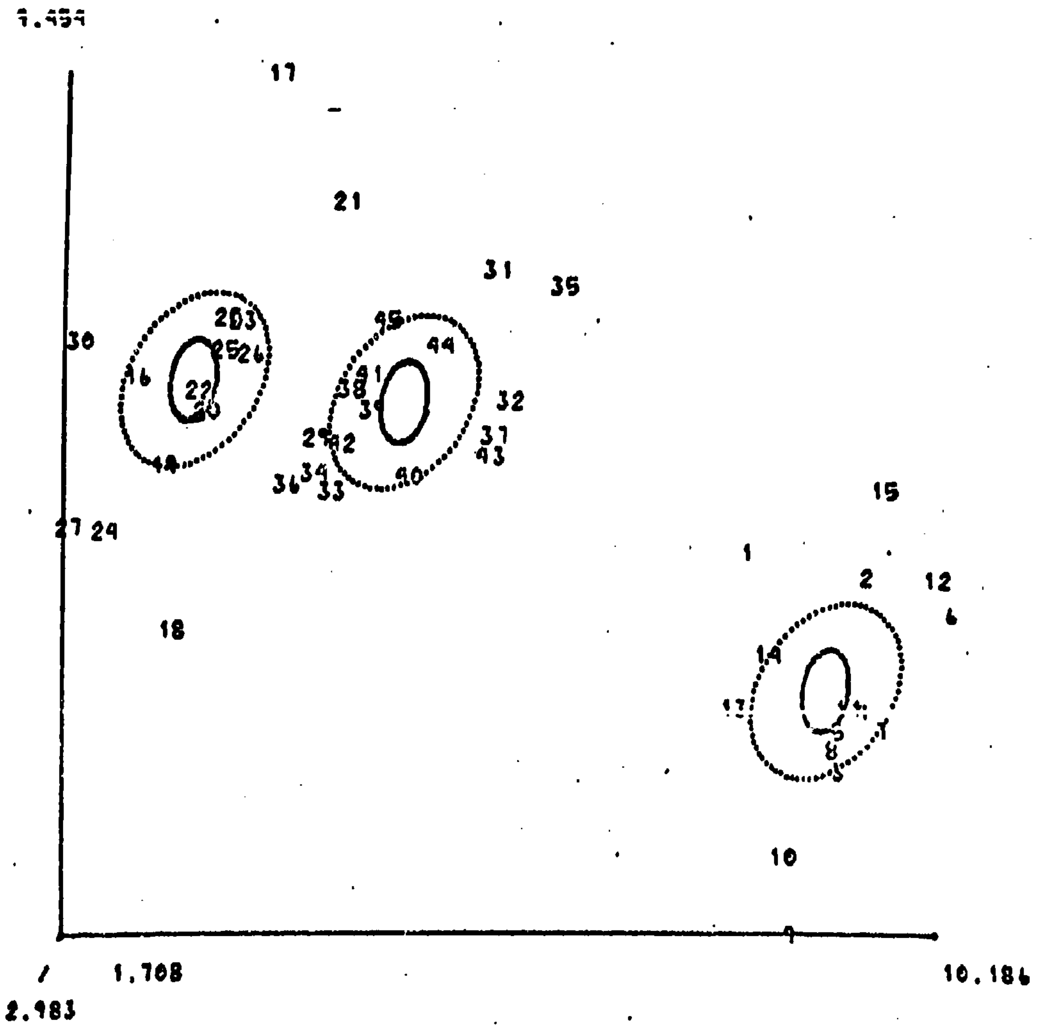
VARIABLE 4AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
1.000 0.0 0.0 0.0



VARIABLE 4AGAINST VECTOR BELOW,(SIMPLE PLANE NO. 3SUPERIMPOSED)
1.000 0.0 0.0 0.0



VARIABLE 4AGAINST VECTOR BELOW.(SIMPLE PLANE NO. 2SUPERIMPOSED)
0.0 0.0 1.000 0.0



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## APPENDIX A

Source Listings for CLUSTR Program

```
LEVEL 19 ( JUNE 70 )                    OS/360  FORTRAN H                         DATE  71.257/21.41.31

         COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=55,SIZE=0000K,
                            SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
  ISN 0002              IMPLICIT REAL*8(D)
  ISN 0003              DOUBLE PRECISION BETAX,DY,ARG,DN,DN,DF,DALF,YCENX,DLOGK,BETAP
  ISN 0004              DOUBLE PRECISION DAR,DRS,DRT
  ISN 0005              DIMENSION SETUP(50),W(1275),FM(1000),F(50),RIN(1275),M(50),D(50),FFCAN COT
                       1G(1000),FRIN(400),NKELI(20),NTEMP(20),KSET(6),FRMT(40)
  ISN 0006              DIMENSION DV(20),W(1275),ICOT(50),IEXT(50),ICL(50),CL(50,20),Z(20)
                       1, VARR(210),CORR(20),FM(20),SPROD(400),KAT(50),INNSUL(50,10)
  ISN 0007              COMMON COSUP(20,20),DXSUP(20),DSPROD(400),X(50,20),NV,NP,NG,
                       1IC(100),NIG(20),NCD(20),NUR(20)
  ISN 0008              REWIND 14
  ISN 0009              WRITE(6,298)
  ISN 0010          298 FORMAT('1')
  ISN 0011              READ(5,50) NP,NV,ALFCOR,ALFEXT,AC2,AE2,AC3,AE3
  ISN 0012           50 FORMAT(I3,1X,I3,6(1X,F4.0))
  ISN 0013              WRITE(6,51) NP,NV
  ISN 0014           51 FORMAT(' NP=',I3,10X,'NV=',I3)
  ISN 0015              READ(5,100) FMT
  ISN 0016          100 FORMAT(20A4)
  ISN 0017              DO 1 I=1,NP
  ISN 0018              READ(5,FMT) (X(I,J),J=1,NV)
  ISN 0019              WRITE(14,105) (X(I,J),J=1,NV)
  ISN 0020          105 FORMAT(1P5E16.6)
  ISN 0021            1 CONTINUE
  ISN 0022              REWIND 14
  ISN 0023              JJJ=0
  ISN 0024          444 JJJ=JJJ+1
  ISN 0025              IF (JJJ.EQ.4) GO TO 811
  ISN 0027              WRITE(6,111) JJJ
  ISN 0028          111 FORMAT('1PASS NO. ',I2)
  ISN 0029              IF(JJJ.EQ.2) ALFCOR=AC2
  ISN 0031              IF(JJJ.EQ.2) ALFEXT=AE2
  ISN 0033              IF(JJJ.EQ.3) ALFCOR=AC3
  ISN 0035              IF(JJJ.EQ.3) ALFEXT=AE3
  ISN 0037              WRITE(6,52) ALFCOR,ALFEXT
  ISN 0038           52 FORMAT(' ALFCOR=',F6.3,5X,'ALFEXT=',F6.3)
  ISN 0039              IF(JJJ.EQ.1) GO TO 445
  ISN 0041              CALL COR2
  ISN 0042              GO TO 446
  ISN 0043          445 CALL COR1
  ISN 0044          446 CONTINUE
  ISN 0045              NVV=NV
  ISN 0046              NVV=(NV*(NV+1))/2
  ISN 0047              WRITE(6,110)
  ISN 0048          110 FORMAT(' MSP IS UPPER TRI PART OF METRIC USED FOR STANDARDIZATION'
                       1)
  ISN 0049              DO 3 I=1,NVV
  ISN 0050              IF(JJJ.EQ.1) GO TO 447
  ISN 0052              DSPROD(I)=DSPROD(I)/(NNIG-NG)
  ISN 0053              GO TO 448
  ISN 0054          447 DSPROD(I)=DSPROD(I)/(NP-1)
  ISN 0055          448 CONTINUE
  ISN 0056            3 WRITE(6,300) I,DSPROD(I)
  ISN 0057          300 FORMAT(' MSP(',I3,')= ',F12.4)
  ISN 0058              CALL DMFSD(DSPROD,NV,1.D-10,IER)
  ISN 0059              DO 4 I=1,NP
ISN 0060          DO 5 J=1,NV
ISN 0061          DDX=X(I,J)
ISN 0062        5 DY(J)=DDX-DXSUM(J)
ISN 0063          CALL DMTDS(DY,1,NV,DSPROC,-1,JER)
ISN 0064          DSUM=0.D0
ISN 0065          DO 6 J=1,NV
ISN 0066        6 DSUM=DSUM+DY(J)**2
ISN 0067          DO 7 J=1,NV
ISN 0068          DPX=DY(J)/DSQRT(DSUM)
ISN 0069          X(I,J)=DPX
ISN 0070        7 CONTINUE
ISN 0071        4 CONTINUE
ISN 0072          DO 10 J=1,NP
ISN 0073          DO 11 I=1,J
ISN 0074          KI=KD(I,J)
ISN 0075          W(KI)=0.0
ISN 0076          DO 12 L=1,NV
ISN 0077       12 W(KI)=W(KI)+X(I,L)*X(J,L)
ISN 0078       11 CONTINUE
ISN 0079       10 CONTINUE
ISN 0080          DO 20 I=1,NP
ISN 0081       20 ICL(I)=0
ISN 0082          DO 21 I=1,NP
ISN 0083          DO 22 J=1,20
ISN 0084       22 CL(I,J)=0.0
ISN 0085       21 CONTINUE
ISN 0086          KCL=1
ISN 0087       97 II=0
ISN 0088          DO 23 I=1,NP
ISN 0089          IEXT(I)=0
ISN 0090       23 ICON(I)=0
ISN 0091          WMAX=0.0
ISN 0092          NPM=NP-1
ISN 0093          DO 24 I=1,NPM
ISN 0094          IPL=I+1
ISN 0095          DO 25 J=IPL,NP
ISN 0096          IF((ICL(I).EQ.1).OR.(ICL(J).EQ.1)) GO TO 25
ISN 0098          IF(W(KD(I,J)).LT.WMAX) GO TO 25
ISN 0100          WMAX=W(KD(I,J))
ISN 0101          I1 = I
ISN 0102          I2 = J
ISN 0103       25 CONTINUE
ISN 0104       24 CONTINUE
ISN 0105          WRITE(6,53) I1,I2,WMAX
ISN 0106       53 FORMAT('OPTS ',I3,' AND ',I3,' ARE CLOSEST WITH CORR ',F12.8)
ISN 0107          RMAX=WMAX
ISN 0108          ALF=FCST(RMAX,2)
ISN 0109          WRITE(6,55) ALF
ISN 0110       55 FORMAT(' ALF= ',F12.8)
ISN 0111          IF(ALF.GT.ALFCCR) GO TO 95
ISN 0113          ICL(I1)=1
ISN 0114          ICL(I2)=1
ISN 0115          ICON(1)=I1
ISN 0116          CL(I1,KCL)=ALF
ISN 0117          CL(I2,KCL)=ALF
ISN 0118          LPT=I2
ISN 0119          LPX=LPT
ISN 0120           ENSC=1.0
ISN 0121          KK=1
ISN 0122       35 KK=KK+1
ISN 0123          WSUM=0.0
ISN 0124          KKM=KK-1
ISN 0125          DO 26 J=1,KKM
ISN 0126       26 WSUM=WSUM+W(KC(ICOR(J),LPT))
ISN 0127           ENSC= ENSC+1.0+2.0*WSUM
ISN 0128          ICOR(KK)=LPT
ISN 0129       36 SUMMAX=-30.0
ISN 0130          DO 29 I=1,NP
ISN 0131          DO 28 J=1,KK
ISN 0132          IF(I.EQ.ICOR(J)) GO TO 29
ISN 0134       28 CONTINUE
ISN 0135          DO 38 J=1,NP
ISN 0136          IF(I.EQ.IEXT(J)) GO TO 29
ISN 0138       38 CONTINUE
ISN 0139          CNSUM=0.0
ISN 0140          DO 31 KX=1,KK
ISN 0141       31 CNSUM=CNSUM+W(KC(ICOR(KX),I))
ISN 0142          IF(CNSUM.LT.SUMMAX) GO TO 32
ISN 0144          SUMMAX=CNSUM
ISN 0145          LPT=I
ISN 0146       32 CONTINUE
ISN 0147       29 CONTINUE
ISN 0148          IF(SUMMAX.LE.0.0) GO TO 56
ISN 0150          RMAX=SUMMAX/SQRT( ENSC)
ISN 0151          KKK=KK+1
ISN 0152          WRITE(6,56) KKK,LPT,RMAX
ISN 0153       56 FORMAT('0POINT ',I3,' IN CLUSTER IS POINT ',I3,',  CORR IS',F8.4)
ISN 0154          ALF=TEST(RMAX,KKK)
ISN 0155          WRITE(6,57) ALF
ISN 0156       57 FORMAT(' ALF= ',F8.4)
ISN 0157          IF(ALF.LE.ALFCOR) GO TO 33
ISN 0159          IF(KK.EQ.2) ICL(LPX)=2
ISN 0161          IF((ALF.GT.ALFCOR).AND.(ALF.LE.ALFEXT)) GO TO 34
ISN 0163          GO TO 95
ISN 0164       33 ICL(LPT)=1
ISN 0165          CL(LPT,KCL)=ALF
ISN 0166          GO TO 35
ISN 0167       34 IF(ICL(LPT).EQ.1) GO TO 37
ISN 0169          ICL(LPT)=2
ISN 0170       37 CONTINUE
ISN 0171          CL(LPT,KCL)=ALF
ISN 0172          IE=IE+1
ISN 0173          IEXT(IE)=LPT
ISN 0174          GO TO 36
ISN 0175       96 KCL=KCL+1
ISN 0176          IF(KCL.EQ.21) GO TO 95
ISN 0178          GO TO 97
ISN 0179       95 CONTINUE
ISN 0180          DO 82 KX=1,2
ISN 0181          JA=10*(KX-1)+1
ISN 0182          JC=JA+9
ISN 0183          WRITE(6,45)
ISN 0184        45 FORMAT('1','PT NO.',32X,'CLUSTERS')
ISN 0185           IF(KX.EQ.2) GO TO 113
ISN 0187           WRITE(6,112)
ISN 0188       112 FORMAT(13X,'NO.  1',4X,'NO.  2',4X,'NO.  3',4X,'NO.  4',4X,
                 1            'NO.  5',4X,'NO.  6',4X,'NO.  7',4X,'NO.  8',4X,
                 2            'NO.  9',4X,'NO. 10')
ISN 0189           GO TO 115
ISN 0190       113 WRITE(6,114)
ISN 0191       114 FORMAT(13X,'NO. 11',4X,'NO. 12',4X,'NO. 13',4X,'NO. 14',4X,
                 1            'NO. 15',4X,'NO. 16',4X,'NO. 17',4X,'NO. 18',4X,
                 2            'NO. 19',4X,'NO. 20')
ISN 0192       115 CONTINUE
ISN 0193           WRITE(6,46)
ISN 0194        46 FORMAT('0')
ISN 0195           DO 47 I=1,NP
ISN 0196        47 WRITE(6,48) I,       (CL(I,J),J=JA,JB)
ISN 0197        48 FORMAT(' ',1X,I3,6X   ,10(2X,F8.6))
ISN 0198        88 CONTINUE
ISN 0199           DO 401 I=1,200
ISN 0200       401 IG(I)=0
ISN 0201           DO 402 I=1,20
ISN 0202       402 NCC(I)=0
ISN 0203           DO 403 J=1,20
ISN 0204           DO 404 I=1,NP
ISN 0205           IF((CL(I,J).GT.0.0).AND.(CL(I,J).LE.ALFCOR)) GO TO 405
ISN 0207           GO TO 404
ISN 0208       405 NCC(J)=NCC(J)+1
ISN 0209       404 CONTINUE
ISN 0210       403 CONTINUE
ISN 0211           DO 407 I=1,20
ISN 0212       407 NCR(I)=0
ISN 0213           DO 409 K=1,20
ISN 0214           NMAX=-1
ISN 0215           DO 406 I=1,20
ISN 0216           DO 408 J=1,20
ISN 0217           IF(I.EQ.NCR(J)) GO TO 406
ISN 0219       408 CONTINUE
ISN 0220           IF(NCC(I).LE.NMAX) GO TO 406
ISN 0222           NMAX=NCC(I)
ISN 0223           IZ=I
ISN 0224       406 CONTINUE
ISN 0225           NCR(K)=IZ
ISN 0226       409 CONTINUE
ISN 0227       411 CONTINUE
ISN 0228           J=0
ISN 0229           KPT=0
ISN 0230       419 J=J+1
ISN 0231           KPT=KPT+1
ISN 0232           IF(NCC(NCR(J)).LE.2) GO TO 99
ISN 0234           ICT=0
ISN 0235           DO 413 NZ=1,NP
ISN 0236           IF(CL(NZ,NCR(J)).GT.0.0) GO TO 414
ISN 0238           GO TO 413
ISN 0239       414 ICT=ICT+1
ISN 0240           IF((CL(NZ,NCR(J)).LE.ALFCOR).AND.(IG(NZ).EQ.0)) IG(NZ)=KPT
ISN 0242       413 CONTINUE
ISN 0243          IF((JJJ.EQ.2).OR.(JJJ.EQ.3)) GO TO 502
ISN 0245          DO 415 JX=1,20
ISN 0246          IMATC=0
ISN 0247          IF(NOR(J).EQ.NOR(JX)) GO TO 415
ISN 0249          DO 416 I=1,NP
ISN 0250          IF((CL(I,NOR(J)).NE.0.0).AND.(CL(I,NOR(JX)).NE.0.0))IMATC=IMATC+1
ISN 0252      416 CONTINUE
ISN 0253          ICTI=ICT-2
ISN 0254          IF((IMATC.GT.2).AND.(IMATC.LE.ICTI)) GO TO 417
ISN 0256          GO TO 415
ISN 0257      417 DO 418 IXX=1,NP
ISN 0258          IF(((CL(IXX,NOR(JX)).LE.ALFCOR).AND.(CL(IXX,NOR(JX)).GT.0.0)).AND.
                 1(IG(IXX).EQ.0)) IG(IXX)=KPI
ISN 0260      418 CONTINUE
ISN 0261      415 CONTINUE
ISN 0262      502 CONTINUE
ISN 0263          NO=0
ISN 0264          DO 420 IQ=1,NP
ISN 0265          IF(IG(IQ).EQ.KPT) NO=NO+1
ISN 0267      420 CONTINUE
ISN 0268          IF(NO.GT.2) GO TO 999
ISN 0270          DO 421 IQ=1,NP
ISN 0271          IF(IG(IQ).EQ.KPT) IG(IQ)=0
ISN 0273      421 CONTINUE
ISN 0274      999 CONTINUE
ISN 0275          GO TO 419
ISN 0276       99 CONTINUE
ISN 0277          DO 425 I=1,20
ISN 0278      425 NIG(I)=0
ISN 0279          DO 422 I=1,20
ISN 0280          DO 423 J=1,NP
ISN 0281          IF(IG(J).EQ.I) NIG(I)=NIG(I)+1
ISN 0283      423 CONTINUE
ISN 0284      422 CONTINUE
ISN 0285          WRITE(6,430) (NIG(I),I=1,20)
ISN 0286      430 FORMAT(' ',20(I2,2X))
ISN 0287          NG=0
ISN 0288          DO 424 I=1,NP
ISN 0289          IF(IG(I).LT.NG) GO TO 424
ISN 0291          NG=IG(I)
ISN 0292      424 CONTINUE
ISN 0293          NNIG=0
ISN 0294          DO 501 I=1,20
ISN 0295      501 NNIG=NNIG+NIG(I)
ISN 0296          WRITE(6,592) NG,NNIG
ISN 0297      592 FORMAT('0NO. OF GROUPS FORMED=',I2,5X,'NO. OF PTS ASSIGNED TO GROU
                 1PS= ',I3)
ISN 0298          DO 951 I=1,NP
ISN 0299      951 READ(14,105) (X(I,J),J=1,NV)
ISN 0300          REWIND 14
ISN 0301          GO TO 444
ISN 0302      811 CONTINUE
ISN 0303          REWIND 15
ISN 0304          CALL COR2
ISN 0305          NVV = (NV*(NV+1))/2
ISN 0306          DO 10000 I=1,NVV
ISN 0307          DSPRCC(I) = CSPROD(I)/(ANIG-NG)
ISN 0308          WRITE (6,30C) I,DSPROD(I)
ISN 03C9    10000 CONTINUE
ISN 0310          DC 11010 I=1,NVV
ISN 0311    11010 VARX(I) = CSPRC (I)
ISN 0312          DO 20CC0 I=1,NG
ISN 0313          IF (NIG(I).EC.0) GO TC 20CC0
ISN 0315          DC 20000 J=1,NV
ISN 0316          DGSUM(I,J) = DGSUM(I,J)/NIG(I)
ISN 0317    20000 CONTINUE
ISN 0318          DC 20C01 I=1,NG
ISN 0319          WRITE(6,11020) (DGSUM(I,J),J=1,NV)
ISN 0320          WRITE(7,11020) (CGSUM(I,J),J=1,NV)
ISN 0321    20001 CONTINUE
ISN 0322    11020 FORMAT(14X,E57.3)
ISN 0323          CALL CNFSO(CSPRCC,NV,1.E-10,IER)
ISN 0324          DC 10C01 I=1,NP
ISN 0325          IF (IC(I).EC.0) GO TC 10003
ISN 0327          KK = IG(I)
ISN 0328          DC 10C02 J=1,NV
ISN 0329          CCX = X(I,J)
ISN 0330    10002 OV(J) = OCX - DGSUM(KK,J)
ISN 0331          GC TO 10005
ISN 0332    10C03 DC 10CC4 J=1,NV
ISN 0333          COX = X(I,J)
ISN 0334    10C04 OV(J) = OCX - DXSUM(J)
ISN 0335    100C5 CALL DMTOS(CV,1,NV,DSPRCC,-1,JER)
ISN 0336          DO 10C07 J=1,NV
ISN 0337          X(I,J) = OV(J)
ISN 0338    10007 CONTINUE
ISN 0339          WRITE(15,2) I,(X(I,J),J=1,NV)
ISN 0340    10C01 CONTINUE
ISN 0341          NVV = (NV*(NV+1))/2
ISN 0342          DC 11000  I=1,NVV
ISN 0343    11000 SPROD(I) = DSPROD(I)
ISN 0344          NVV = NV**2
ISN 0345          REWIND 15
ISN 0346          KFACT = NV
          C       USE WITH SUBROUTINES FACT,WRIR,LTERM,INSLD                  FCAN 006
ISN 0347          NIN=5                                                       FCANA006
ISN 0348          NOUT=6                                                      FCANB006
ISN 0349          CALF = 0.01C0
          C       LIMITATION... 50 VARIABLES,  20 FACTORS                     FCAN 009
          C       READ HEADER CARD                                            FCAN 011
ISN 0350          DO 17 KARD = 1,NP
ISN 0351          READ (15,2) IC,(SETUP(J),J=1,KFACT)
ISN 0352        2 FORMAT(1X,I2,3X,7F16.7)
ISN 0353          DO 13 J=1,KFACT
ISN 0354          IX = (J-1)*NP + ID
ISN 0355       13 FM(IX) = SETUP(J)
ISN 0356       17 CONTINUE
ISN 0357          DC 14 K=1,NF
ISN 0358          H(K) =0.
ISN 0359          DO 14 J=1,KFACT
ISN 0360          KX = (J-1)*NP + K
ISN 0361       14 H(K) = H(K) + FM(KX)**2
ISN 0362          DO 1002  J#1,NP
ISN 0363          IF%H%J<< 1003,1003,1004                                   FCAN 195
ISN 0364     1003 D%J< # 0.                                                 FCAN 196
ISN 0365          GO TO 1002                                                FCAN 197
ISN 0366     1004 D%J< # SQRT %H%J<<                                        FCAN 198
ISN 0367     1002 CONTINUE                                                  FCAN 199
ISN 0368          NSOL # 0                                                  FCAN 200
ISN 0369          CM = DFLOAT(KFACT-1)/2.D0
ISN 0370          SNTHTA = DSQRT(BETAP(6.D0/DFLOAT(NP-KFACT+1),0.500,DM))
ISN 0371          IF (SNTHTA.LT.0.1) SNTHTA = 0.1
ISN 0373          DO 1001 I#1,NP                                            FCAN 201
ISN 0374          IF%H%I%< 1001,1001,1005                                   FCAN 202
ISN 0375     1005 ICO # 0                                                   FCAN 203
ISN 0376          DO 1006 K # 1,KFACT                                       FCAN 204
ISN 0377          IALP # %K-1<*NP & I                                       FCAN 205
ISN 0378     1006 FRIH%K< # FH%IALP</D%I<                                   FCAN 206
ISN 0379          BND # 8.                                                  FCAN 207
ISN 0380     1021 DO 1005 J#1,NP                                            FCAN 208
ISN 0381          SUM & 0.                                                  FCAN 209
ISN 0382          DO 1007 K#1,KFACT                                         FCAN 210
ISN 0383          IALP # %K-1<*NP & J                                       FCAN 211
ISN 0384     1007 SUM # SUM & FH%IALP<*FRIH%K<                              FCAN 212
ISN 0385          FC%J< # SUM                                               FCAN 213
ISN 0386          JN: # NP & J                                              FCAN 214
ISN 0387          IF%ABS%FG%J<< .GE. 1.E8 * D%J< <  GO TO 1011
ISN 0389          CRT = ABS(AINT(SNGL(10.*FG(J)/D(J))*0.1/SNTHTA))
ISN 0390          IF%CRT - 8A%< 1010,1011,1011                              FCAN 216
ISN 0391     1011 FC%JNT< # 0.                                              FCAN 217
ISN 0392          GO TO 1008                                                FCAN 218
ISN 0393     1010 FC%JNT< # SQRT %BND - CRT<                                FCAN 219
ISN 0394     1008 CONTINUE                                                  FCAN 220
ISN 0395          IF%IGO<  1080,1080,2008                                   FCAN 221
ISN 0396     1080 NCZE # 0                                                  FCAN 222
ISN 0397          DO 1081 J#1,NP                                            FCAN 223
ISN 0398          JNT # NP&J                                                FCAN 224
ISN 0399          IF%FG%JNT<< 1081,1082,1081                                FCAN 225
ISN 0400     1082 NCZE # NCZE & 1                                           FCAN 226
ISN 0401     1081 CONTINUE                                                  FCAN 227
ISN 0402          IF%NCZE - 2< 1083,1083,2008                               FCAN 228
ISN 0403     1083 BND # BND - 1.                                            FCAN 229
ISN 0404          GO TO 1021                                                FCAN 230
ISN 0405     2008 IF%IGO - 7< 2013,1020,1020                                FCAN 231
ISN 0406     2013 TNOR # 0.                                                 FCAN 232
ISN 0407          DO 1013 K#1,KFACT                                         FCAN 233
ISN 0408          AV # 0.                                                   FCAN 234
ISN 0409          ASQ # 0.                                                  FCAN 235
ISN 0410          DO 1014 J#1,NP                                            FCAN 236
ISN 0411          JALP # %K-1<*NP & J                                       FCAN 237
ISN 0412          JNT # NP & J                                              FCAN 238
ISN 0413          AV # AV & FH%JALP<*FG%J<*FG%JNT<                          FCAN 239
ISN 0414     1014 ASQ # ASQ & FH%JALP<*FH%JALP<*FG%JNT<                     FCAN 240
ISN 0415          IF %ASQ .EQ. 0.< GO TO 1013
ISN 0417          E # AV/ASQ                                                FCAN 241
ISN 0418          FRIH%K< #%FRIH%K< - E</%1. - E*FRIH%K<<                   FCAN 242
ISN 0419     1013 TNOR # TNOR & FRIH%K<*FRIH%K<                             FCAN 243
ISN 0420          TNOR # SQRT %TNOR<                                        FCAN 244
ISN 0421          DO 1016 K#1,KFACT                                            FCAN 245
ISN 0422     1016 FRIM(K< # FRIM(K</TNOR                                       FCAN 246
ISN 0423          IGO # IGO & 1                                                FCAN 247
ISN 0424          GO TO (1017,1018,1019,1019,1019,1019,1019<,IGO              FCAN 248
ISN 0425     1017 BND # 3.                                                     FCAN 249
ISN 0426          GO TO 1021                                                   FCAN 250
ISN 0427     1018 BND # 2.                                                     FCAN 251
ISN 0428          GO TO 1021                                                   FCAN 252
ISN 0429     1019 BND # 1.                                                     FCAN 253
ISN 0430          GO TO 1021                                                   FCAN 254
ISN 0431     1020 NCON # 0                                                     FCAN 255
ISN 0432          VMAX = 0.0
ISN 0433          VMIN = 0.0
ISN 0434          DO 1121 J#1,NP                                               FCAN 256
ISN 0435          IF(C(J<< 1121,1121,1022                                      FCAN 257
ISN 0436     1022 RAT(J) = FC(J)/C(J)
ISN 0437          IF (ABS(RAT(J))-SNTHTA) 1023,1023,1121
ISN 0438     1023 NCON # NCON & 1
ISN 0439          IF (VMAX.LE.RAT(J)) VMAX = RAT(J)
ISN 0441          IF (VMIN.GE.RAT(J)) VMIN = RAT(J)
ISN 0443          RAT(J) = 0.0
ISN 0444     1121 CONTINUE
ISN 0445          IF (NCON.LE.KFACT+2) GO TO 1001
ISN 0447          DELTA = (ARSIN(VMAX)-ARSIN(VMIN))/2.0
ISN 0448          ARG = (DSIN(DELTA))**2
ISN 0449          CN # DFLOAT(KFACT-1</2.D0
ISN 0450          DY = BETAX(ARG,0.5D0,CN)
ISN 0451          DM # DFLOAT(NCON-KFACT&1<
ISN 0452          DN # DFLOAT(NP-NCON&1<
ISN 0453          DP # BETAX(DY,DM,DN<
ISN 0454          IF(DP.GT.DALF< GO TO 1001
ISN 0456     1322 IF(NSOL< 1024,1024,1025                                      FCAN 262
ISN 0457     1025 DO 1026 L#1,NSOL                                             FCAN 263
ISN 0458          SUM # 0.                                                     FCAN 264
ISN 0459          DO 1027 K#1,KFACT                                            FCAN 265
ISN 0460          IAL # (L-1<*KFACT & K                                        FCAN 266
ISN 0461     1027 SUM # SUM & FRIM(K<*RIN(IAL<                                 FCAN 267
ISN 0462          IF (ABS(SUM)-0.7) 1026,1001,1001
ISN 0463     1026 CONTINUE                                                     FCAN 269
ISN 0464     1024 DO 1030 K#1,KFACT
ISN 0465          IEL # NSOL*KFACT & K                                         FCAN 271
ISN 0466     1030 RIN(IEL< # FRIM(K<                                           FCAN 272
ISN 0467          NSOL # NSOL & 1                                              FCAN 273
ISN 0468          WRITE (NOUT,1031< NSOL,ACON,DP
ISN 0469     1031 FORMAT(1H0//13H  PLANE NO. V I3,20H  OF SIMPLE STRUCTURE WITH I4,FCAN 275
                 135H ZERO LOADINGS (-.1 LESS V/M LESS (.1<,D12.5/<
ISN 0470          CALL WRIT(FC,NP,0<                                           FCAN 277
ISN 0471          WRITE(NOUT,1033< NSOL                                        FCAN 278
ISN 0472     1033 FORMAT(1H0// 75H  VECTOR WHICH TRANSFORMS ORIGINAL FACTOR MATRIX IFCAN 279
                 1NTO THE ABOVE PLANE NO. V I3<                                FCAN 280
ISN 0473          CALL WRIT(FRIM,KFACT,0<                                      FCAN 281
ISN 0474          CALL MTDS (FRIM,1,NV,SPROD,-2,JER)
ISN 0475          SS = 0.
ISN 0476          DO 740 JJ=1,NV
ISN 0477      740 SS = SS + FRIM(JJ)**2
ISN 0478          DO 741 JK = 1,NV
```

```
  741 SETUP(JK) = FRIP(JK)/SQRT(SS)
      WRITE (6,790)
  790 FORMAT(1H0,10X,'TRANSFORMATION VECTOR TO TRANSFORM RAW DATA TO
     1SIMPLE STRUCTURE')
      CALL WRIR(SETUP,KFACT,0)
      WRITE (7,11002) (SETUP(JL),JL=1,KFACT)
      CALL PPPD (FRIM,VARX,Z,1,NV,0,1,NV)
      DO 11003 LL=1,NV
      K = (LL*(LL+1))/2
11003 CORR(LL) = Z(LL)/SQRT(VARX(K))
      WRITE (6,11004) (CORR(LL),LL=1,NV)
11004 FORMAT(1H0,3X,'CORRELATIONS BETWEEN ORIGINAL VARIABLES AND THIS
     1VECTOR ARE',//10F12.4)
      DO 11005 LL=1,NP
      INNSCL(LL,NSCL) = 0
      IF (RAT(LL).GE.0.0) INNSCL(LL,NSCL) = 1
11005 CONTINUE
 1001 CONTINUE
      IF(NSCL< 1640,1640,1641
 1640 WRITE(NOUT,1642<
 1642 FORMAT(1H1 21X 25HNO SIMPLE STRUCTURE SOLUTIONS<
      GO TO 1643
 1641 DO 1070 J#1,NSCL
      DO 1070 I#1,J
      SUM # 0.
      DO 1071 K#1,KFACT
      KI # (I-1<*KFACT & K
      KJ # (J-1<*KFACT & K
 1071 SUM # SUM & RIN(KI<*RIN(KJ<
      IJ = ND(I,J)
 1070 R(IJ< # SUM
      WRITE(NOUT,1073<
 1073 FORMAT(1H1 18X 33HCOSINES BETWEEN REFERENCE VECTORS<
      CALL WRIR(R,NSCL,1<
      WRITE(NOUT,1600<
 1600 FORMAT(1H1 27X 18HSUMMARY OF RESULTS/<
      KAE # (NSCL-1</6 & 1
      IE #  MOD (NSCL,6<
      DO 1602 KA#1,KAE
      IF(KA-KAE< 1604,1606,1606
 1604 KE#6
      GO TO 1608
 1606 IF(IE< 1604,1604,1610
 1610 KE#IE
 1608 DO 1612 KB#1,KBE
 1612 KSET(KB< # KB & 6*(KA-1<
      WRITE(NOUT,1614< (KSET(J<,J#1,KBE<
 1614 FORMAT(1H0/10H VAR      V I2, 8H        V I2, 8H        V I2, 8H
     1  V I2, 8H        V I2, 8H        V I2, 1H /<
      DO 1616 KC#1,NP
      DO 1618 KD#1,KBE
      SUM # 0.
      DO 1620 KG#1,KFACT
      KRI # (KG-1<*NP & KC
      KIJ = ((KA-1)*6 + KD - 1)*KFACT + KG
 1620 SUM # SUM & FRIM(KRI<*RIN(KIJ<
 1618 SETUP(KC1) = SUM
      WRITE(NOUT,1624) KC,(SETUP(J),J=1,K6)
 1624 FORMAT(1H I3,6F10.4)
 1616 CONTINUE
 1602 CONTINUE
      DO 1644 I=1,NP
      READ (14,105)(X(I,J),J=1,NV)
 1644 CONTINUE
      REWIND 14
      IF (NSOL.GE.10) GO TO 1646
      L = NSOL + 1
      DO 1645 I=L,10
      DO 1645 KK=1,NP
      INNSOL(KK,I) = 0
 1645 CONTINUE
 1646 DO 1647 I=1,NP
      WRITE (7,1650) I,IG(I),(INNSOL(I,J),J=1,10),(X(I,J),J=1,NV)
 1647 CONTINUE
 1650 FORMAT(1X,I2,I1,10I1,8F7.3)
      WRITE (6,1780)
 1780 FORMAT(1H1)
      WRITE (6,1790)
 1790 FORMAT(41X,'PLANES')
      WRITE (6,1791)
 1791 FORMAT(10X,'POINT #',1X,'CLUSTER #',5X,'1',1X,'2',1X,'3',1X,'4',1X
     1,'5',1X,'6',1X,'7',1X,'8',1X,'9',1X,'10',10X,'COORDINATES(OBSERVAT
     2IONS)')
      DO 1710 I=1,NP
      WRITE (6,1792) I,IG(I),(INNSOL(I,J),J=1,10),(X(I,J),J=1,NV)
 1710 CONTINUE
 1792 FORMAT(14X,I2,6X,I1,10X,10(I1,1X),5X,8(F7.3,2X))
 1643 STOP
      END
```

```
LEVEL 19 ( JUNE 70 )                          OS/360  FORTRAN H                          DATE  71.257/21.42.06

         COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=58,SIZE=0000K,
                            SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
  ISN 0002              FUNCTION KD(I,J)
  ISN 0003              IF(I.LT.J) GO TO 1
  ISN 0005              KD=(I*(I-1))/2+J
  ISN 0006              GO TO 99
  ISN 0007            1 KD=(J*(J-1))/2+I
  ISN 0008           99 RETURN
  ISN 0009              END
```

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LEVEL 19 ( JUNE 70 )                      OS/360  FORTRAN H                                   DATE  71.257/21.42.11

        COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=58,SIZE=0000K,
                           SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
                  CWRIR                                                                        042
                  C      SUBROUTINE FOR WRITING CORRELATIONS, MEANS, AND FACTOR LOADINGS       043
  ISN 0002               SUBROUTINE WRIR%R,NV,NN<                                              044
                  C      NN IS ZERO FOR FACTOR LOADINGS AND MEANS, ONE OTHERWISE               045
  ISN 0003               NIN#5                                                                A045
  ISN 0004               NOUT#6                                                               B045
  ISN 0005               DIMENSION R%1275<,JWR%10<,RWR%10<                                     046
  ISN 0006               WRITE%NOUT,1183<                                                      047
  ISN 0007               DO 1101 I#1,NV                                                        048
  ISN 0008               NRE # %NV-1</10 & 1
  ISN 0009               DO 1102 J#1,NRE                                                       050
  ISN 0010               IF%10*J-NV< 1103,1103,1104
  ISN 0011          1103 K#10
  ISN 0012               GO TO 1105                                                            053
  ISN 0013          1104 K#10-10*J&NV
  ISN 0014          1105 DO 1107 L#1,K                                                         055
  ISN 0015          1107 JWR%L< # 10*%J-1<&L
  ISN 0016               WRITE%NOUT,1181< %JWR%M<,M#1,K<                                       057
  ISN 0017          1181 FORMAT%1H0 13X I2,9%9X,I2<<
  ISN 0018               IF%NN< 1129,1130,1129                                                 059
  ISN 0019          1130 DO 1131 M#1,K                                                         060
  ISN 0020               MF # JWR%M<                                                           061
  ISN 0021          1131 RWR%M< # R%MF<                                                        062
  ISN 0022               WRITE%NOUT,1185< %RWR%N<,N#1,K<                                       063
  ISN 0023          1185 FORMAT%7X 10F11.5<
  ISN 0024               GO TO 1102                                                            065
  ISN 0025          1129 DO 1120 M#1,K                                                         066
  ISN 0026               MR = KC(I,JWR(M))
  ISN 0027          1120 RWR%M< # R%MR<                                                        068
  ISN 0028               WRITE%NOUT,1182< I,%RWR%N<,N#1,K<                                     069
  ISN 0029          1182 FORMAT%5H ROW ,I2,10F11.5<
  ISN 0030          1102 CONTINUE                                                              071
  ISN 0031               IF%NN< 1127,1128,1127                                                 072
  ISN 0032          1127 WRITE%NOUT,1183<                                                      073
  ISN 0033          1183 FORMAT%1H0<                                                           074
  ISN 0034          1101 CONTINUE                                                              075
  ISN 0035          1128 RETURN                                                                076
  ISN 0036               END                                                                   077
```

NOT REPRODUCIBLE

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         COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=58 SIZE=0000K,
                            SOURCE,BCD,NOLIST,NODECK,LOAD MAP,NOEDIT,NOID,NOXREF
  ISN 0002              SUBROUTINE COR1
  ISN 0003              IMPLICIT REAL*8(D)
  ISN 0004              COMMON DCSUM(20,20),DXSUM(20),DSPROD(400),X(50,20),NV,NP,NG,
                       1IG(100),NIG(20),NCO(20),NCR(20)
  ISN 0005              NC=(NV*(NV+1))/2
  ISN 0006              DO 1 I=1,NV
  ISN 0007            1 DXSUM(I)=0.D0
  ISN 0008              DO 2 I=1,NC
  ISN 0009            2 DSPROD(I)=0.D0
  ISN 0010              DO 3 I=1,NV
  ISN 0011              DO 4 J=1,NP
  ISN 0012              DX=X(J,I)
  ISN 0013            4 DXSUM(I)=DXSUM(I)+DX
  ISN 0014              DXSUM(I)=DXSUM(I)/(DFLOAT(NP))
  ISN 0015              WRITE(6,200) I,DXSUM(I)
  ISN 0016          200 FORMAT(' MEAN OF VAR(',I2,')= ',F9.4)
  ISN 0017            3 CONTINUE
  ISN 0018              WRITE(6,300)
  ISN 0019          300 FORMAT(' DSPROD IS UPPER TRI PART OF MATRIX OF CORRECTED S.S. AND
                       1S.P. FOR NV VARS. STORED BY COLS')
  ISN 0020              DO 5 J=1,NV
  ISN 0021              DO 6 I=1,J
  ISN 0022              KI=KD(I,J)
  ISN 0023              DO 7 L=1,NP
  ISN 0024              DX1=X(L,J)
  ISN 0025              DX2=X(L,I)
  ISN 0026            7 DSPROD(KI)=DSPROD(KI)+DX1*DX2
  ISN 0027              DSPROD(KI)=DSPROD(KI)-DXSUM(I)*DXSUM(J)*DFLOAT(NP)
  ISN 0028              WRITE(6,100) KI,DSPROD(KI)
  ISN 0029          100 FORMAT(' DSPROD(',I3,')= ',F13.4)
  ISN 0030            6 CONTINUE
  ISN 0031            5 CONTINUE
  ISN 0032              RETURN
  ISN 0033              END
```

```
        COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=5.,SIZE=0000K,
                              SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
  ISN 0002            SUBROUTINE COR2
  ISN 0003            IMPLICIT REAL*8(D)
  ISN 0004            COMMON DGSUM(20,20),DXSUM(20),DSPROD(400),X(50,20),NV,NP,NG,
                     1IG(100),NIG(20),KCO(20),NCR(20)
  ISN 0005            NC=(NV*(NV+1))/2
  ISN 0006            DO 2 I=1,NC
  ISN 0007          2 DSPROD(I)=0.D0
  ISN 0008            DO 11 I=1,NG
  ISN 0009            DO 12 J=1,NV
  ISN 0010         12 DGSUM(I,J)=0.D0
  ISN 0011         11 CONTINUE
  ISN 0012            DO 13 K=1,NG
  ISN 0013            DO 3 I=1,NV
  ISN 0014            DO 4 J=1,NP
  ISN 0015            DX=X(J,I)
  ISN 0016            IF(IG(J).NE.K) GO TO 4
  ISN 0018            DGSUM(K,I)=DGSUM(K,I)+DX
  ISN 0019          4 CONTINUE
  ISN 0020          3 CONTINUE
  ISN 0021         13 CONTINUE
  ISN 0022            DO 57 I=1,NV
  ISN 0023            DO 58 K=1,NG
  ISN 0024            WRITE(6,159) I,K,DGSUM(K,I)
  ISN 0025        159 FORMAT(' SUM OF OBSERS. FOR VAR',I2,' IN GROUP',I2,'= ',D12.4)
  ISN 0026         58 CONTINUE
  ISN 0027         57 CONTINUE
  ISN 0028            WRITE(6,300)
  ISN 0029        300 FORMAT(' DSPROD IS UPPER TRI PART OF MATRIX OF CORRECTED S.S. AND
                     1S.P. FOR NV VARS, STORED BY-COLS')
  ISN 0030            DO 5 J=1,NV
  ISN 0031            DO 6 I=1,J
  ISN 0032            KI=KO(I,J)
  ISN 0033            DO 7 L=1,NP
  ISN 0034            IF(IG(L).EQ.0) GO TO 7
  ISN 0036            DX1=X(L,J)
  ISN 0037            DX2=X(L,I)
  ISN 0038            DSPROD(KI)=DSPROD(KI)+DX1*DX2
  ISN 0039          7 CONTINUE
  ISN 0040            DCOR=0.D0
  ISN 0041            DO 14 L=1,NG
  ISN 0042            IF(NIG(L).EQ.0) GO TO 23
  ISN 0044            DCOR=DCOR+DGSUM(L,J)*DGSUM(L,I)/NIG(L)
  ISN 0045            GO TO 14
  ISN 0046         23 NC=NC-1
  ISN 0047         14 CONTINUE
  ISN 0048            DSPROD(KI)=DSPROD(KI)-DCOR
  ISN 0049            WRITE(6,100) KI,DSPROD(KI)
  ISN 0050        100 FORMAT(' DSPROD(',I3,')= ',F13.4)
  ISN 0051          6 CONTINUE
  ISN 0052          5 CONTINUE
  ISN 0053            RETURN
  ISN 0054            END
```

NOT REPRODUCIBLE

```
LEVEL 19 ( JUNE 70 )                    OS/360  FORTRAN H                              DATE  71.257/21.42.25

          COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=58,SIZE=0000K,
                               SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
  ISN 0002             FUNCTION TEST(RMAX,K)
  ISN 0003             IMPLICIT REAL*8(D)
  ISN 0004             DOUBLE PRECISION BETAX
  ISN 0005             DOUBLE PRECISION YORMX
  ISN 0006             DOUBLE PRECISION DLGGM
  ISN 0007             COMMON DGSUM(20,20),DXSUM(20),DSPROD(400),X(50,20),NV,NP,NG,
                      1IC(100),NIG(20),NCO(20),NCR(20)
  ISN 0008             CRMAX=RMAX
  ISN 0009             CRMAX=1.D0-CRMAX**2
  ISN 0010             DK=K
  ISN 0011             DV=NV
  ISN 0012             DV=(DV-1.D0)*.5D0
  ISN 0013             DN=NP
  ISN 0014             DNN=DN-DK+1.D0
  ISN 0015             DPM=.5D0*BETAX(CRMAX,DV,.5D0)
  ISN 0016             DKK=DK-1.D0
  ISN 0017             DTEST=BETAX(DPM,DKK,DNN)
  ISN 0018             TEST=DTEST
  ISN 0019             RETURN
  ISN 0020             END
```

NOT REPRODUCIBLE

```
          COMPILER OPTIONS - NAME=  MAIN,OPT=02,LINECNT=58,SIZE=0000K,
                             SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
               C     BETAP REQUIRES FUNCTION SUBROUTINES BETAX,DLGGM, AND YCRMX
ISN 0002               DOUBLE PRECISION FUNCTION BETAP%DP,DM,DN<
ISN 0003                IMPLICIT REAL *8%D,B<
ISN 0004               DIMENSION CARG%4<,DFUN%4<
ISN 0005               NOUT # 6
ISN 0006                 DZ#0.D0
ISN 0007                DU#1.D0
ISN 0008               IF%DP*%DU-DP<<  95,91,20
ISN 0009         20    IF%%DM.LE.DZ<.OR.%DN.LE.DZ<<  GO TO 95
ISN 0011               IF%DN.EQ.DU<  GO TO 90
ISN 0013               IF %DN.EQ.DL<  GO TC 92
ISN 0015               DL # DZ
ISN 0016               DIF # DU/3.D0
ISN 0017               DLX # -DP
ISN 0018               DUX # DU - DP
ISN 0019               JJ # 0
ISN 0020         89    JEND # 3
ISN 0021         86    JJ # JJ + 1
ISN 0022               DO 80 J#1,JEND
ISN 0023               DAP # %DU+DL</2.D0
ISN 0024               IF%%DU-DL<  .LT.1.D-40< GO TO 1
ISN 0026               IF%%%DU-DL</DP.LT.1.D-12<.AND.%DL.GT.1.D-12<<  GO TO 195
ISN 0028               DO 81 I#1,2
ISN 0029               CARG%I< # DL + %DU-DL<*DIF*DFLOAT%I<
ISN 0030               DFUN%I< # BETAX%CARG%I<,DM,DN< - DP
ISN 0031               IF.DFUN%I<.EQ.DZ< CMP # CARG%I<
ISN 0033               IF%DFUN%I<< 81,1,82
ISN 0034         82    DU # CARG%I<
ISN 0035               DUX # DFUN%I<
ISN 0036               IF%I.EQ.1< GO TC 80
ISN 0038               DL # CARG%I<
ISN 0039               DLX # DFUN%I<
ISN 0040               GO TO 80
ISN 0041         81    CONTINUE
ISN 0042               DL # CARG%2<
ISN 0043               DLX # DFUN%2<
ISN 0044         80    CONTINUE
ISN 0045               JEND # 2
ISN 0046               CMP #%DU+DL</2.D0
ISN 0047               DFD # DUX - DLX
ISN 0048               IF%%DFD.LT.1.D-10 <.AND.%DFD/DP.LT.1.D-6<< GO TO 1
ISN 0050               DECR # DUX*%DU-DL</DFD
ISN 0051               DMP # DU - DECR
ISN 0052               IF%%DMP - DL<.LT.1.D-40<  GO TO 195
ISN 0054               IF%%%DMP - DL<.LT.1.D-12<.AND.%DL.GT.1.D-12<< GO TO 195
ISN 0056               DFUN%3< # BETAX%DMP,DM,DN< - DP
ISN 0057               DABF # DABS%DFUN%3<<
ISN 0058               DFUN3    # DFUN%3<
ISN 0059               IF%%DABF.LT.1.D-10<.AND.%DABF/DP.LT.1.D-6<<  GO TO 1
ISN 0061               IF %DMP.LT.1.D-40< GO TC 195
ISN 0063               IF(((DL-DMP).LT.1.D-12).AND.(DU.GT.0.9999999999DO)) GO TO 195
ISN 0065               IF (DFUN(3)) 83,1,84
ISN 0066         83    IF%DECR.LE.0.5D0*%DU-DL<< GO TO 185
```

NOT REPRODUCIBLE

```
ISN CC68            OMPU # OMP
ISN CC69            OMP # 5.CO*%DMP-DL< E DL
ISN CC70            DFUNE    # BETAX%CMP,DM,DN< - DP
ISN CC71            IF%CFUNE  < 183,1,40
ISN CC72      4C    DU # CMP
ISN CC73            DUX # DFUNE
ISN CC74            DL # CMPU
ISN CC75            DLX # DFUN%3<
ISN CC76            IF%JJ-10< 88,89,195
ISN CC77      84    IF%DECR.GE.C.1CO*%DU-DL<<  GO TO 184
ISN CC79            DMPU # CMP
ISN CC80            CMP # CU - 5.CO*DEC
ISN CC81            CFUNE    # BETAX%CMP,DM,DN< - DP
ISN CC82            IF%CFUNE  <  41,1,184
ISN CC83      41    DU # CMPU
ISN CC84            DUX # DFUN%3<
ISN CC85            DL # CMP
ISN CC86            DLX # CFUNE
ISN CC87            IF%JJ-10< 88,89,195
ISN CC88      183   DL # CMP
ISN CC89            DLX # DFUNE
ISN CC90            IF%JJ-10< 88,89,195
ISN CC91      184   DL # CMP
ISN CC92            DLX # DFUNE
ISN CC93            IF%JJ-10< 88,89,195
ISN CC94      1     BETAP # DMP
ISN CC95            RETURN
ISN CC96      195   DRES # CFUNE   E DP
ISN 0097            WRITE%NOUT,196< CP,DM,DN,DMP,DRES
ISN CC98      196   FORMAT%1HO,5X,43HAO CONVERGENCE IN BETAP IN DOUBLE PRECISION /
                    11X,9HINPUT P =  D16.10,4H M =  D18.10,4H N = C18.10,9H LAST X =
                    2D16.10,13H PRODUCES P = C18.10<
ISN CC99            GC TO 1
ISN C100      91    DMP # DP
ISN C1C1            GC TO 1
ISN 0102      95    DMP # DZ
ISN 0103            WRITE %NOUT,1C1< CP,DM,DN
ISN C1C4      1C1   FORMAT %1HC,28HARGUMENTS FOR BETAP WERE P =  C12.4, 4H M =  D12.4,
                    14H N =  D12.4, 28H RESULT HAS BEEN SET TO ZERO <
ISN C1C5            GC TO 1
ISN C1C6      90    DMP # DU - %DL-CP<**%DU/DN<
ISN C107            GC TO 1
ISN C108      92    DMP # DP**%CU/DN<
ISN C1C9            GO TO 1
ISN 0110             END
```

```
LEVEL 19 ( JUNE 70 )                         OS/360  FORTRAN H                              DATE  71.257/21.42.31

          COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=56,SIZE=0000K,
                             SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
 ISN 0002            DOUBLE PRECISION FUNCTION YGRMX(DZ)
 ISN 0003            IMPLICIT  REAL*8(D)
 ISN 0004            DOUBLE PRECISION YGRMX
 ISN 0005            DPI = .3989422804D1433
 ISN 0006            DX = DABS(DZ)
 ISN 0007            IF(DX.GT.3.D0) GO TO 10
 ISN 0009            DAL = 0.D0
 ISN 0010            DBL = 1.D0
 ISN 0011            DAH = DX
 ISN 0012            DBH = 1.D0
 ISN 0013            DAN = 0.D0
 ISN 0014       5    DAN = DAN + 1.D0
 ISN 0015            DAI = -(2.D0*DAN - 1.D0) *DX*DX
 ISN 0016            DBI = 4.D0*DAN - 1.D0
 ISN 0017            DAL = DBI*DAH + DAI*DAL
 ISN 0018            DBL = DBI*DBH + DAI * DBL
 ISN 0019            DAI = DX*DX - DAI
 ISN 0020            DBI = 2.D0 + DBI
 ISN 0021            DAH = DBI*DAL + DAI*DAH
 ISN 0022            DBH = DBI*DBL + DAI*DBH
 ISN 0023            DFA = DAL/DBL
 ISN 0024            DFB = DAH/DBH
 ISN 0025            IF(DFB.EQ.0.D0) GO TO 20
 ISN 0027            IF(DABS((DFB-DFA)/DFA).LE.1.D-14) GO TO 20
 ISN 0029            GO TO 5
 ISN 0030      10    DAL = 0.D0
 ISN 0031            DBL = 1.D0
 ISN 0032            DAH = 1.D0
 ISN 0033            DBH = DX
 ISN 0034            DBI = DX
 ISN 0035            DAN = 1.D0
 ISN 0036            DFA = 1.D0/DX
 ISN 0037      15    DAN = DAN + 1.D0
 ISN 0038            DAI = DAN - 1.D0
 ISN 0039            DAC = DBI*DAH + DAI*DAL
 ISN 0040            DBC = DBI*DBH + DAI*DBL
 ISN 0041            DFB = DAC/DBC
 ISN 0042            DAL = DAH
 ISN 0043            DBL = DBH
 ISN 0044            DAH = DAC
 ISN 0045            DBH = DBC
 ISN 0046            IF(DFB.EQ.0.D0) GO TO 20
 ISN 0048            IF(DABS((DFB-DFA)/DFB).LE.1.D-14) GO TO 20
 ISN 0050            DFA = DFB
 ISN 0051            GO TO 15
 ISN 0052      20    YGRMX = DPI*DFB*DEXP(-DX*DX/2.D0)
 ISN 0053            IF(DX.LE.3.D0) YGRMX = 0.5D0 - YGRMX
 ISN 0055            IF(DZ.GT.0.D0) YGRMX = 1.D0 - YGRMX
 ISN 0057            RETURN
 ISN 0058            END
```

```
LEVEL 19 ( JUNE 70 )                    OS/360  FORTRAN H                         DATE  71.257/21.42.43

       COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=58,SIZE=0000K,
                          SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
 ISN 0002                  DOUBLE PRECISION FUNCTION DLGGM(DX)
 ISN 0003                  IMPLICIT REAL *8(D)
 ISN 0004                  DY=DX
 ISN 0005                  DTERM=1.D0
 ISN 0006                  IF(DX)1,1,2
 ISN 0007         1        DLGGM=0.D0
 ISN 0008                  RETURN
 ISN 0009         2        IF(DY-18.D0) 3,3,4
 ISN 0010         3        DTERM=DTERM*DY
 ISN 0011                  DY=DY+1.D0
 ISN 0012                  GOTO2
 ISN 0013         4        DLGGM= (DY-.5D0)* DLOG(DY)-DY+1.D0/(12.D0*DY)-1.D0/(360.D0*DY**3)
                          1+ 1.D0/ (1260.D0*DY**5) -1.D0/ (1680.D0*DY**7 )+.918938533204673D0
                          2  -DLOG(DTERM)
 ISN 0014                  RETURN
 ISN 0015                  END
```

```
LEVEL 19 ( JUNE 70 )                    OS/360  FORTRAN H                         DATE  71.257/21.42.47

          COMPILER OPTIONS - NAME=  MAIN,OPT=00,LINECNT=58,SIZE=0000K,
                             SOURCE,BCD,NOLIST,NODECK,LOAD,MAP,NOEDIT,NOID,NOXREF
  ISN 0002              DOUBLE PRECISION FUNCTION BETAX(DX,DM,DN)
  ISN 0003              IMPLICIT REAL *8( D,Y,W)
  ISN 0004              DTEST=.99999999999D0
  ISN 0005              DZ=0.D0
  ISN 0006              DE= 1.D0
  ISN 0007             DX = DX - DE
  ISN 0008              DSUM=DZ
  ISN 0009              IF( DX.LE. DZ) GOTO1
  ISN 0011              IF( DN.LE. DZ) GOTO2
  ISN 0013              IF( DX*(DE-DX) .LT. DZ) GOTO1
  ISN 0015              IF( DX.GT.DTEST) GOTO2
  ISN 0017             IF (DX.LT.1.D-42) GO TO 3
  ISN 0019             DCUB=DMIN1(200D.D0,30.D0/(DX*(1.D0-DX)))
  ISN 0020              IFLAG=0
  ISN 0021             IF (DX.LE.DM/(DM+DN)) GO TO 101
  ISN 0023              DX=DE-DX
  ISN 0024              DHLD=DN
  ISN 0025              DN=DM
  ISN 0026              DM=DHLD
  ISN 0027             DX = DM - DE
  ISN 0028              IFLAG=1
  ISN 0029       101    IF(DM.GE. 15.D0) GOTO7
  ISN 0031             IF(DM+DN.GE.DCUB)GO TO 700
  ISN 0033              I=DM+.5D0
  ISN 0034              DNT=I
  ISN 0035              IF( DABS(DM-DNT) .LT. 1.D-7) GOTO4
  ISN 0037              LI= 20.D0-DM+DE
  ISN 0038              DK=DM-DE
  ISN 0039              DLGN=DLGGM(DN)
  ISN 0040              DNX=DN*DLOG( DE-DX)
  ISN 0041             DLDX = DLOG(DX)
  ISN 0042              DO 5 I=1,LI
  ISN 0043              DK=DK+DE
  ISN 0044             IF (I.GE.2) GO TO 305
  ISN 0046              DT=DLGGM(DN+DK)- DLGN-DLGGM(DK+DE)+DK*DLOG(DX)+DNX
  ISN 0047              IF(DT.GT. -60.D0) GOTO6
  ISN 0049             GO TO 5
  ISN 0050        305  DT = DT + DLOG((DN + DK)/(DK + DE)) + DLDX
  ISN 0051             IF (DT.GT.-60.D0) GO TO 6
  ISN 0053       5      CONTINUE
  ISN 0054              GOTO7
  ISN 0055       6      DSUM=DEXP(DT)
  ISN 0056              DTERM=DSUM
  ISN 0057              LI= 20.D0-DK
  ISN 0058              DO 8 I=1,LI
  ISN 0059              DTERM=( DN+DK)*DX*DTERM/(DK+DE)
  ISN 0060              IF(DTERM/DSUM.LT. 1.D-12) GOTO8
  ISN 0062              DSUM=DSUM+DTERM
  ISN 0063       8      DK=DK+DE
  ISN 0064            7 IF(DX+DM.GT.DCUB)GO TO 700
  ISN 0066             DCPT = DZ
  ISN 0067             DSML = DSUM
  ISN 0068             DK = DK + DE
```

NOT REPRODUCIBLE

```
ISN 0069            DSN = DE
ISN 0070            DAL = DZ
ISN 0071            DBL = DE
ISN 0072            DAM = DE
ISN 0073            DBM = DE
ISN 0074            DCFLT = DLGGM(DK+DN) - DLGGM(DK+DE) - DLGGM(DN) + DK*DLOG(DX) +
                   1DN * DLOG(DE - DX)
ISN 0075            DCFLU = DCFLT
ISN 0076            KFLAG = 1
ISN 0077            DSUM = DE
ISN 0078            KFLAG = 1
ISN 0079            DO 135 KK = 1,80
ISN 0080            DKK = KK
ISN 0081            DD1 = DK + DKK - DE
ISN 0082            DD2 = DK + 2.D0*DKK - DE
ISN 0083            DCN1 = -DX*DD1*(DD1+DN)/(DD2*(DD2-DE))
ISN 0084            DCN2 = DX*DKK*(DN-DKK)/(DD2*(DD2+DE))
ISN 0085            DAL = DAM + DCN1*DAL
ISN 0086            DAM = DAL + DCN2*DAM
ISN 0087            DBL = DBM + DCN1*DBL
ISN 0088            DBM = DBL + DCN2*DSN
ISN 0089            IF(DABS(DBM).LT.1.D-60) GO TO 201
ISN 0091            IF((DCFLU.LT.-80.D0).OR.(DCFLU.GT.80.D0)) GO TO 201
ISN 0093            IF(KFLAG.EQ.0) GO TO 203
ISN 0095            DCFT = DEXP(DCFLT)*DSN
ISN 0096            KFLAG = 0
ISN 0097            DSUM = DCFT
ISN 0098            KFLAG = 0
ISN 0099      203   DCFU = DSUM
ISN 0100            DSUM = DCFT*DAM/DBM
ISN 0101            IF(DDM2.EQ.DZ) GO TO 208
ISN 0103            IF (DSUM) 135,208,202
ISN 0104      202   IF(DABS(DSUM-DCFU)/DSUM.LT.1.D-12) GO TO 208
ISN 0106            GO TO 135
ISN 0107      201   IF(DSN.EQ.DZ) GO TO 135
ISN 0109            KFLAG = 1
ISN 0110            DCFLU = DCFLT + DLOG(DABS(DAM)) - DLOG(DABS(DBM)) + DLOG(DSUM)
ISN 0111            DSN = DSIGN(DE,DAM)*DSIGN(DE,DBM)*DSN*DSIGN(DE,DSUM)
ISN 0112      135   CONTINUE
ISN 0113      208   IF(KFLAG.EQ.1) DSUM = DZ
ISN 0115            DSUM = DSUM + DSML
ISN 0116            GO TO 8
ISN 0117     700     DTH=DE/3.D0
ISN 0118             DX=DK*DE
ISN 0119             DW1=(DN*DX)**DTH
ISN 0120             DW2= (DK*(DE-DX))**DTH
ISN 0121             DU= 3.D0*(DW1*(DE-DE/(9.D0*DN))-DW2*(DE-DE/(9.D0*DK)))/
                   1  DSQRT( DW1*DW1/DN+DW2*DW2/DK)
ISN 0122             DSUM=DSUM+YORNX(DU)
ISN 0123     8       BETAX=DSUM
ISN 0124             IF( IFLAG.EQ. 0) RETURN
ISN 0126             BETAX=DE-BETAX
ISN 0127             DX=DE-DX
ISN 0128             DN=DN
ISN 0129             DN=DHLD
ISN 0130             RETURN
```

NOT REPRODUCIBLE

```
ISN 0131      4     DA = DNT
ISN C132            GC TO 7
ISN 0133      1      WRITE(6,100) CM,CN,CX
ISN C134    100      FURMAT(///5X, 4CHERROR IA INPUT PARAMETER BETAX SET TO 0. ,
                  1 /5X, 2HM=, D20.8,2HN=,D20.8,2HX=,D20.8)
ISN C135      3      EETAX=CZ
ISN 0136             RETURN
ISN C137      2      EETAX=CE
ISN C138             RETURN
ISN 0139             ENC
```

## APPENDIX B

## Source Listings for ELLIPSE Program

```
FORTRAN IV G LEVEL  19                  MAIN                 DATE = 71257          14/31/15

              C THIS IS THE MAIN PROGRAM
 0001               COMMON GDSX,GDSY,GTEXT,CPOINT,GDSE,GDSER,X(50,8),NV,NP,NGG,NSOL,
                   1NC,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
                   2R(16),RWRKNG(16),GINPUT,RINPUT(8),SUG(10,8),NNSOL(50,2),IGWORK(50
                   3),NIG(5),DSPROD(36),IG(50),DCSUM(5,8),DXSUM(8),AWRKNG(40)
 0002               CALL CALC
 0003               CALL EXIBIT
 0004               STOP
 0005               END
```

```
FORTRAN IV G LEVEL  19                  KD                 DATE = 71257          14/31/24

                 FUNCTION KD(I,J)
                 IF(I.LT.J) GO TO 1
                 KD=(I*(I-1))/2+J
                 GO TO 99
               1 KD=(J*(J-1))/2+I
              99 RETURN
                 END
```

NOT REPRODUCIBLE

```
FORTRAN IV G LEVEL  19                    CALC                 DATE = 71257          14/31/45

                SUBROUTINE CALC
                COMMON GDSX,GDSY,GTEXT,CPOINT,GDSE,GDSER,X(50,9),NV,NP,KGG,NSOL,
               1NG,CLMEAN(5,8),A(36,11),XMAX(9),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
               2R(15),RWAKNG(16),GINPUT,XINPUT(8),SUG(10,8),INNSOL(50,2),IGWORK(50
               3),NIG(5),DSPROD(36),IG(50),DSSUM(5,9),DXSUM(8),RANKG(40)
                READ (5,2) NP,NV,NG,NSOL
                NVV = (NV*(NV+1))/2
              2 FORMAT(4(1X,I3))
                DO 14 I=1,NG
             14 READ (5,451) (CLMEAN(I,J),J=1,NV)
            451 FORMAT (14X,6F7.3)
                DO 114 I=1,NSOL
            114 READ (5,451) (SUG(I,J),J=1,NV)
           1650 FORMAT(1X,I2,I1,I20,8F7.3)
                DO 10 I=1,NP
                READ (5,1650) INNSOL(I,2),IG(I),INNSOL(I,1),(X(I,J),J=1,NV)
             10 CONTINUE
             13 DO 425 I=1,NG
            425 NIG(I) = 0
                DO 422 I=1,NG
                DO 423 J=1,NP
                IF (IG(J).EQ.I) NIG(I) = NIG(I) + 1
            423 CONTINUE
            422 CONTINUE
                NNIG = 0
                DO 429 I=1,NG
            429 NNIG = NNIG + NIG(I)
                DO 428 I=1,NP
                IGWORK(I) = IG(I)
            428 CONTINUE
                NGG = NG
                CALL CORZ
                DO 434 I=1,NVV
            434 DSPROD(I) = DSPROD(I)/(NNIG-NG)
                CALL SINV(DSPROD,NV,1.E-7,IER)
                WRITE (6,1661) IER
           1661 FORMAT (1X,'ERROR CODE AFTER CALL TO SINV IS',I3)
                DO 435 I=1,NVV
                A(I,1) = DSPROD(I)
            435 CONTINUE
                IF (NSOL.EQ.0) GO TO 450
                DO 455 I=1,NV
                DXSUM(I) = 0.0
                DO 456 J=1,NP
            456 DXSUM(I) = DXSUM(I) + X(J,I)
                DXSUM(I) = DXSUM(I)/NP
            455 CONTINUE
                DO 426 I=1,NSOL
                DO 427 J=1,NP
                IGWORK(J) = 0
                ICX = INNSOL(J,1)/10**(10-I)
                ICY = ICX/10
                IF (10*ICY.NE.ICX) IGWORK(J) = IG(J)
            427 CONTINUE
                DO 432 K=1,NG
            432 NIG(K) = 0
```

NOT REPRODUCIBLE

```
          DO 430 K=1,NG
          DO 431 J=1,NP
          IF (IGWORK(J) .EQ. K) NIG(K) = NIG(K) + 1
      431 CONTINUE
      430 CONTINUE
          NNIG = 0
          DO 441 K=1,NG
      441 NNIG = NNIG + NIG(K)
          CALL CORIS
          DO 442 K=1,NVV
      442 DSPROD(K) = DSPROD(K)/NNIG
          CALL SINV(DSPROD,NV,1.E-7,IER)
          WRITE (6,1661) IER
          DO 433 J=1,NVV
          A(J,I+1) = DSPROD(J)
      433 CONTINUE
      426 CONTINUE
      450 DO 15 I=1,NV
          XMAX(I) = X(1,I)
          XMIN(I) = X(1,I)
          DO 15 J=1,NP
          IF (X(J,I).GT.XMAX(I)) XMAX(I) = X(J,I)
          IF (X(J,I).LT.XMIN(I)) XMIN(I)=X(J,I)
       15 CONTINUE
          WRITE (6,1651) XMAX(1)
     1651 FORMAT(1X,'XMAX(1)=',F7.3)
          RETURN
          END
```

NOT REPRODUCIBLE

```
FORTRAN IV G LEVEL  19                 CORIS               DATE = 71257          14/32/03

      SUBROUTINE CORIS
      COMMON GDSX,GDSY,GTEXT,GPOINT,GDSE,GDSER,X(50,8),NV,NP,NGG,NSOL,
     1NG,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
     2R(16),RWRKNG(16),GINPUT,RINPUT(8),SUG(10,8),INNSCL(50,2),IGWORK(50
     3),NIG(5),DSPROD(36),IG(50),DGSUM(5,8),DXSUM(8),AWRKNG(40)
      DO 10 I=1,NV
      DO 11 J=1,I
      KI = KB(I,J)
      DSPROD(KI) = 0.0
      DO 12 L=1,NP
      IF (IGWORK(L).EQ.0) GO TO 12
      IF (IG(L).EQ.0) GO TO 7
      K = IG(L)
      DX1 = X(L,J) - CLMEAN(K,J)
      DX2 = X(L,I) - CLMEAN(K,I)
      DSPROD(KI) = DSPROD(KI) + DX1*DX2
      GO TO 12
    7 DX1 = X(L,J) - DXSUM(J)
      DX2 = X(L,I) - DXSUM(I)
      DSPROD(KI) = DSPROD(KI) + DX1*DX2
   12 CONTINUE
      WRITE (6,100) KI,DSPROD(KI)
  100 FORMAT(' DSPROD(',I3,')= ',F13.4)
   11 CONTINUE
   10 CONTINUE
      RETURN
      END
```

```
FORTRAN IV G LEVEL  19            COR2                DATE = 71257          14/32/11

            SUBROUTINE COR2
            COMMON GCSX,GDSY,GTEXT,GPOINT,GDSE,GDSER,X(50,8),NV,NP,NGG,NSOL,
           1NG,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
           2R(16),RANKNG(16),GINPUT,RINPUT(8),SUG(10,8),INNSCL(50,2),IGWORK(50
           3),NIG(5),DSPROD(36),IG(50),DGSUM(5,8),DXSUM(8),ARRANG(40)
            NC=(NV*(NV+1))/2
            DO 2 I=1,NC
          2 DSPROD(I)=0.D0
            DO 11 I=1,NGG
            DO 12 J=1,NV
         12 DGSUM(I,J)=0.D0
         11 CONTINUE
            DO 13 K=1,NGG
            DO 3 I=1,NV
            DO 4 J=1,NP
            DX=X(J,I)
            IF(IGWORK(J).NE.K) GO TO 4
            DGSUM(K,I)=DGSUM(K,I)+DX
          4 CONTINUE
          3 CONTINUE
         13 CONTINUE
            DO 57 I=1,NV
            DO 58 K=1,NGG
            WRITE(6,159) I,K,DGSUM(K,I)
        159 FORMAT(' SUM OF CASERS. FOR VAR',I2,'IN GROUP',I2,'= ',D12.4)
         58 CONTINUE
         57 CONTINUE
            WRITE(6,300)
        300 FORMAT(' DSPROD IS UPPER TRI PART OF MATRIX OF CORRECTED S.S. AND
           1S.P. FOR NV VARS, STORED BY COLS')
            DO 5 J=1,NV
            DO 6 I=1,J
            KI=KO(I,J)
            DO 7 L=1,NP
            IF(IGWORK(L).EQ.0) GO TO 7
            DX1=X(L,J)
            DX2=X(L,I)
            DSPROD(KI)=DSPROD(KI)+DX1*DX2
          7 CONTINUE
            DCOR=0.D0
            DO 14 L=1,NGG
            IF(NIG(L).EQ.0) GO TO 23
            DCOR=DCOR+DGSUM(L,J)*DGSUM(L,I)/NIG(L)
            GO TO 14
         23 NGG = NGG-1
         14 CONTINUE
            DSPROD(KI)=DSPROD(KI)-DCOR
            WRITE(6,100) KI,DSPROD(KI)
        100 FORMAT(' DSPROD(',I3,')= ',F13.4)
          6 CONTINUE
          5 CONTINUE
            RETURN
            END
```

```
FORTRAN IV G LEVEL  19              EXIBIT              DATE = 71257          14/32/21

                   SUBROUTINE EXIBIT
                   COMMON GDSX,GDSY,GTEXT,GPOINT,GDSE,GDSER,X(50,8),NV,NP,KCC,NSOL,
                  1NG,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
                  2R(16),RWRKNG(16),GINPUT,RINPUT(8),SUG(10,4),INKSOL(50,2),IGWORK(50
                  3),NIG(5),DSPROD(36),IG(5)),DSSUM(5,8),DKSUM(8),AWRKNG(40)
                   CALL DISPLA(GDSX,GDSY,GTEXT,GPOINT,GDSE,GDSER,GINPUT)
                   WRITE (6,1654)
              1654 FORMAT(1X,'DISPLA SUCCESSFUL')
                   I = LIGHTS(1,1,-NV,27,-29,31)
                   CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
                   CALL MESSGE
                   WRITE (4,1655)
              1655 FORMAT('P1FOLLOWING ANY DISPLAY OF PROJECTIONS PRESS'/
                  1'PCKEY 29 TO CHANGE THE VECTOR FORMING ONE OF THE AXES'/
                  2'PGAND KEY 31 TO CHANGE THE VARIABLE FORMING ANOTHER AXIS.'/
                  3'PONOW PRESS ANY KEY TO GO ON.')
                   CALL WRFMTS(GTEXT)
                   LP = PLOT(GTEXT)
                   LP = DETAIN(NDET)
                   LP = UNPLOT(GTEXT)
                   CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
                   NTEST = 1
                   NGO = 0
                20 CALL KEYIN (NAXIS,NTEST)
                   IF (NAXIS.EQ.30)GO TO 40
                   CALL ELLPSE (NAXIS)
                16 KK = DETAIN(NDET)
                   KK = NDET(4)
                17 IF (KK.EQ.30) GO TO 40
                   NTEST = KK
                   IF (KK.EQ.29) GO TO 80
                   IF (KK.EQ.31) GO TO 80
                   IF (KK.LE.NSOL) GO TO 61
                   LP = UNPLOT(GTEXT)
                   CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
                   WRITE (4,62)
                62 FORMAT('P1ERROR OR SINGULAR SOLUTION - PRESS ANOTHER KEY'/
                  1'P IF TRAPPED,PRESS KEY 29')
                   CALL WRFMTS(GTEXT)
                   LP = PLOT(GTEXT)
                   GO TO 16
                80 LP = UNPLOT(GDSX,GDSY,GPOINT,GTEXT)
                   CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
                   LP = UNPLOT(GDSE)
                   CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
                   GO TO 20
                61 CALL RELPSE (KK,NAXIS)
                   IF (KK.NE.-15) GO TO 103
                   KK = 15
                   GO TO 17
               103 KK = DETAIN(NDET)
                   KK = NDET(4)
                   LP = UNPLOT(GDSER)
                   CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
                   GO TO 17
                40 CALL BLANK
```

```
FORTRAN IV G LEVEL  19                    EXIBIT              DATE = 71257          14/32/21

 0050               CALL RESET(GDSX,GDSY,GDSE,GDSER,GTEXT,GPOINT,GINPUT)
 0051               RETURN
 0052               END
```

```
FORTRAN IV G LEVEL  19                MESSGE              DATE = 71257          14/32/29

0001              SUBROUTINE MESSGE
0002              COMMON GDSX,GDSY,GTEXT,GPOINT,GDSE,GDSER,X(50,8),NV,NP,NGG,NSOL,
                 1NG,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
                 2R(16),RANKNG(16),GINPUT,RINPUT(S),SUG(10,8),INNSCL(50,2),IGWORK(50
                 3),NIG(5),DSPROD(36),IG(50),DGSUM(5,8),DXSUM(8),AWRKNG(40)
0003              WRITE (4,460)
0004          460 FORMAT('P1THIS PROGRAM DISPLAYS CLUSTERS BY PROJECTING THEM ON'/
                 1'PCVARIOUS 2-DIMENSIONAL SUBSPACES.SIMPLE STRUCTURE'/
                 2'POSOLUTIONS CAN ALSO BE INDICATED.REFER TO YOUR'/
                 3'POINSTRUCTION CHART.HAVE YOU ENTERED ALL NECESSARY'/
                 4'PODATA VIA IEBGENER?'/
                 5'PC',10X,'THE BOTTOM ROW OF THE PROGRAM '/
                 6'PCFUNCTION KEYS IS LIT UP.THEY WILL BE USED'/
                 7'POAS DIRECTED.THE DARK ONE,KEY NO. 30,IS'/
                 8'PCTHE PANIC BUTTON.IT WILL RETURN YOU TO THE '/
                 9'PCMONITOR.'/'PLOIN CASE OF PANIC PRESS KEY 30'/
                 1'PEONOW PRESS ANY KEY TO GET STARTED')
0005              CALL WRFMT$(GTEXT)
0006              LP = PLOT(GTEXT)
0007              WRITE (6,1653) LP
0008         1653 FORMAT(1X,'LP AFTER FIRST MESSAGE IS',I4)
0009              I = DETAIN(NDET)
0010              I = UNPLOT(GTEXT)
0011              CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
0012              RETURN
0013              END
```

NOT REPRODUCIBLE

```
      SUBROUTINE KEVIN (KAX S,NTEST)
      COMMON GDSX,GDSY,GTEX ,GPOINT,GDSE,GDSER,X(50,6),NV,NP,NCG,NSOL,
     1NG,CLMEAN(5,8),A(36,1 ),XMAX(3),XMIN(3),XLO,YLO,XHI,YHI,NDET(5),
     2R(16),RANKNG(16),GINPUT,RINPUT(8),SUG(10,8),INNSOL(50,2),IGWORK(50
     3),NIG(5),DSPROD(36),I(50),DSSUM(5,8),DXSUM(8),AWANKG(40)
      IF (NTEST.EQ.31) GO TO 101
   99 CALL BLANK
      CALL RCURS
      CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
      WRITE (4,190) NSOL
  190 FORMAT('P1'/'P0    ACCORDING TO THE ROTATION PROGRAM THE'/
     1'P FOLLOWING VECTORS TRANSFORM RAW DATA INTO'/
     2'P SIMPLE STRUCTURE SOLUTIONS NO.1 TO',I3)
      CALL WRFMTS(GINPUT)
      DO 110 J=1,NSOL
      WRITE (4,191) (SUG(J,K),K=1,NV)
  191 FORMAT('P  ',8F5.3)
      CALL WRFMTS(GINPUT)
      IDUM = PLOT(GINPUT)
      CALL RESET(GINPUT,GTEXT)
  110 CONTINUE
      WRITE (4,3000)
 3000 FORMAT('P0ENTER A TRANSFORMATION VECTOR.'/
     1'P NO MORE THAN 7 CHARACTERS,DECIMAL POINT'/
     2'P MUST BE TYPED'/
     3'P DEPRESS JUMP KEY AFTER EACH COORDINATE'/
     5'P X(1)='/'U.',10X/
     6'P X(2)='/'U.',10X/
     7'P X(3)='/'U.',10X/
     8'P X(4)='/'U.',10X/
     9'P X(5)='/'U.',10X/
     1'P X(6)='/'U.',10X/
     2'P X(7)='/'U.',10X/
     3'P X(8)='/'U.',10X/
     4'P IF PREVIOUS VECTOR OK,JUST PRESS EOB')
      CALL WRFMTS(GTEXT)
      J = PLOT(GTEXT)
      CALL CURSS(GTEXT,126)
      L = DETAIN (NOLT)
      CALL SCTSW(GTEXT)
      CALL DVTOM(GTEXT)
      DO 211 JI = 1,NV
  211 RWRKNG(JI) = RINPUT(JI)
      READ (4,3005) (RINPUT(I),I=1,NV)
 3005 FORMAT( /////G10.4//G10.4//G10.4//G10.4//G10.4//G10.4//G10.4//G10.
     14)
      DO 213 IJ = 1,NV
      IF (RINPUT(IJ).NE.0.0) GO TO 214
  213 CONTINUE
      IF (NGG.EQ.0) GO TO 214
      DO 216 IJ = 1,NV
  216 RINPUT(IJ) = RWRKNG(IJ)
  214 CALL BUFRS
      LQ = UNPLOT(GINPUT)
      LP = UNPLOT(GTEXT)
      CALL BLANK
```

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FORTRAN IV G LEVEL  19                KEYIN              DATE = 71257          14/32/38

           NGG = 1
           CALL RCURS
           CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
       101 WRITE (4,1658)
      1653 FORMAT('P1 PRESS ONE KEY CORRESPONDING TO THE AXIS'/
          1'P - VARIABLE YOU WISH TO SELECT, OR 30 IF'/
          2'P  YOU WISH TC STCP.IF YOU WISH TO MAKE '/
          3'P CHANGES IN YOUR VECTOR PRESS KEY 28.')
           CALL WRFMT$ (GTEXT)
           LP = PLCT(GTEXT)
        60 NAXIS = DETAIN(NDET)
           NAXIS = NDET(4)
           IF (NAXIS.EQ.30) RETURN
           IF (NAXIS.EQ.28) GO TO 99
           IF (NAXIS.LE.NV) GC TO 54
           LP = UNPLCT(GTEXT)
           CALL RESET(GTEXT,GCSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
           WRITE(4,52)
        52 FORMAT('P1 ERROR - PRESS PROPER KEY')
           CALL WRFMT$(GTEXT)
           LP = PLCT(GTEXT)
           GO TO 60
        54 LP = UNPLCT(GTEXT)
           CALL RESET(GTEXT,GCSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
           RETURN
           END
```

NOT REPRODUCIBLE

```
FORTRAN IV G LEVEL  19              ELLPSE              DATE = 71257          14/32/48

                 SUBROUTINE ELLPSE (NAXIS)
                 DIMENSION ELPSX(5,72),ELPSY(5,72),U(400),V(100)
                 COMMON GDSX,GDSY,GTEXT,IPOINT,GDSE,GDSEN,X(50,8),NV,NP,NCG,NSOL,
                1NG,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NDET(5),
                2R(16),AWRANG(16),GINPUT,RINPUT(6),SUG(10,8),INNSOL(50,2),IGWORK(50
                3),AIG(5),DSPROD(36),IC(10),DGSUM(5,8),DXSUM(8),AWRANG(40)
                 EQUIVALENCE (IGWORK(1),ELPSX(1,1),V(1)),(ELPSY(1,1),U(1))
                 DO 10 K=1,NV
              10 R(K) = 0.0
                 R(NAXIS) = 1.0
                 NVV = NV*2
                 KK = NV + 1
                 DO 11 K=KK,NVV
              11 R(K) = RINPUT(K-NV)
                 R(NV+NAXIS) = 0.0
                 SS = 0.0
                 DO 20 K= KK,NVV
                 SS = SS + R(K)**2
              20 CONTINUE
                 IF (SS.GT.0.0) GO TO 211
             213 WRITE (4,1659)
            1659 FORMAT('P1',5X,'INCONSISTENT.PRESS KEY 29 AND NEW VECTOR')
                 CALL WRPTS (GTEXT)
                 LP = PLOT(GTEXT)
                 RETURN
             211 DO 21 K=KK,NVV
                 R(K) = R(K)/SQRT(SS)
              21 CONTINUE
                 NVV = (NV*(NV+1))/2
                 DO 12 K=1,NVV
              12 AWRANG(K) = A(K,1)
                 CALL MPRD(AWRANG,R,RWRKPG,NV,NV,1,0,2)
                 CALL GTPRD(R,RWRANG,AWRANG,NV,2,2)
                 SII = AWRANG(1)
                 SIJ = AWRANG(2)
                 SJJ = AWRANG(4)
                 DO 13 K=1,NV
                 DO 13 KK = 1,NP
              13 U((K-1)*NP+KK) = X(KK,K)
                 CALL GMPRD (U,R,V,NP,NV,2)
                 DO 14 K=1,NP
                 U(K) = V(K)
              14 V(K) = V(NP+K)
                 VMAX = V(1)
                 VMIN = V(1)
                 DO 15 K=2,NP
                 IF (V(K).GT.VMAX) VMAX = V(K)
                 IF (V(K).LT.VMIN) VMIN = V(K)
              15 CONTINUE
                 XLO = (-0.2*XMAX(NAXIS)+1.8*XMIN(NAXIS))/1.6
                 XHI = (1.8*XMAX(NAXIS)-0.2*XMIN(NAXIS))/1.6
                 YLO = (-0.2*VMAX+1.8*VMIN)/1.6
                 YHI = (1.8*VMAX-0.2*VMIN)/1.6
                 CALL UCGRD (XLO,YLO,XHI,YHI)
                 CALL PLACE(GDSX,XMIN(NAXIS),VMAX)
                 CALL LINE(GDSX,XMIN(NAXIS),VMIN)
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NOT REPRODUCIBLE

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FORTRAN IV G LEVEL  19              ELLPSE              DATE = 71257          14/32/48

                CALL LINE(GDSX,XMAX(NAXIS),VMIN)
                LP = PLOT(GDSX)
                CALL PLACE(GDSY,XLO,0.5*(YHI+VMAX))
                WRITE (4,120) VMAX
            120 FORMAT('P8.',F9.3)
                CALL WRFMTS(GDSY)
                CALL PLACE(GDSY,XLO,(2.*YLO+VMIN)/3.)
                WRITE (4,120) VMIN
                CALL WRFMTS(GDSY)
                XLX = (XLO+6.*XMIN(NAXIS))/7.
                YLX = (YLO+2.*VMIN)/3.
                CALL PLACE(GDSY,XLX,YLX)
                WRITE (4,121) XMIN(NAXIS)
            121 FORMAT('P6./',F9.3)
                CALL WRFMTS(GDSY)
                XLXX = (9.*XMAX(NAXIS)-2.*XHI)/7.
                CALL PLACE(GDSY,XLXX,YLX)
                WRITE (4,120) XMAX(NAXIS)
                CALL WRFMTS(GDSY)
                LP = PLOT(GDSY)
                DO 110 K=1,NP
                CALL PLACE(GPOINT,U(K),V(K))
                WRITE (4,111) INXSCL(K,2)
            111 FORMAT ('P8.',I2)
                CALL WRFMTS(GPOINT)
            110 CONTINUE
                LP = PLOT(GPOINT)
                ANGLE = 1.0
                DEN = (SII-SJJ)**2 + 4.*SIJ**2
                IF (DEN.NE.0.0) ANGLE = SQRT(0.5+SQRT(0.25-SIJ**2/DEN))
                AXIS1 = SII*ANGLE**2+SJJ*(1.-ANGLE**2)+2.*ANGLE*SQRT(1.-ANGLE**2)*
               1SIJ
                IF (AXIS1.LE.0.0) GO TO 213
                AXIS1 = 1./SQRT(AXIS1)
                AXIS2 = (1. - ANGLE**2)*SII+ANGLE**2*SJJ
               1-2.*ANGLE*SQRT(1.-ANGLE**2)*SIJ
                IF (AXIS2.LE.0.0) GO TO 213
                AXIS2 = 1./SQRT(AXIS2)
                SIG = SIGN(1.,(AXIS2-AXIS1)*SIJ)
                C = 3.141593/180.
                THETA = 5.0
                DO 31 K = 1,NV
                DO 31 KK = 1,NG
                AWRANG ((K-1)*NG+KK) = CLMEAN (KK,K)
             31 CONTINUE
                CALL GMPRD (AWRANG,R,RWRANG,NG,NV,2)
                DO 150 K=1,NG
                DO 150 KK =1,72
                RADIAN = THETA*C*KK
                ELX = AXIS1*COS(RADIAN)
                ELY = AXIS2*SIN(RADIAN)
                ELPSX(K,KK) = RWRANG(K) + ELX*ANGLE - ELY*SQRT(1.-ANGLE**2)*SIG
                ELPSY(K,KK) = RWRANG(NG+K)+ELX*SQRT(1.-ANGLE**2)*SIG + ELY*ANGLE
            150 CONTINUE
                DO 100 K=1,NG
                DO 100 KK=1,72
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NOT REPRODUCIBLE

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FORTRAN IV G LEVEL  19                  ELLPSE             DATE = 71257          14/32/48

 0108                CALL POINT(GDSE,ELPSX(K,KK),ELPSY(K,KK))
 0109            100 CONTINUE
 0110                LP = PLOT(GDSE)
 0111                KK = NV + 1
 0112                NVV = NV*2
 0113                WRITE (4,1657)NAXIS, (R(K),K=KK,NVV)
 0114           1657 FORMAT('P1',10X,'VARIABLE',I2,'AGAINST VECTOR'/
                    2'P  ',8F8.3)
 0115                CALL WRFMTS(STEXT)
 0116                LP = PLOT(GTEXT)
 0117                RETURN
 0118                END
```

NOT REPRODUCIBLE

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FORTRAN IV G LEVEL  19              RELPSE              DATE = 71257          14/33/01

              SUBROUTINE RELPSE(KK,NAXIS)
              DIMENSION RELPSX(5,72),RELPSY(5,72)
              COMMON GDSX,GDSY,GTEXT,GPOINT,GDSE,GDSER,X(50,8),NV,NP,NGG,KSOL,
             1NG,CLMEAN(5,8),A(36,11),XMAX(8),XMIN(8),XLO,YLO,XHI,YHI,NOCT(5),
             2R(16),ARRKNG(16),GINPUT,RINPUT(8),SUG(10,8),IANSCL(53,2),ICHOSK(53
             3),NIG(5),DSPROD(36),IG(50),DCSUM(5,8),DXSUM(8),ARRKNG(40)
              KX = NV + 1
              KY = 2*NV
              DO 212 JX = KX,KY
              IF(R(JX)**2.GT.0.0) GO TO 213
          212 CONTINUE
              GO TO 211
          213 CALL UCCRD (XLO,YLO,XHI,YHI)
              NVV = (NV*(NV+1))/2
              DO 61 K=1,NVV
              ARRKNG(K) = A(K,KK+1)
           61 CONTINUE
              CALL MPRD(ARRKNG,R,RARKNG,NV,NV,1,0,2)
              CALL GTPRD(R,RARKNG,ARRKNG,NV,2,2)
              SII = ARRKNG(1)
              SIJ = ARRKNG(2)
              SJJ = ARRKNG(4)
              ANGLE = 1.0
              DEN = (SII-SJJ)**2 + 4.*SIJ**2
              IF (DEN.NE.0.0) ANGLE = SQRT(0.5+SQRT(0.25-SIJ**2/DEN))
              AXIS1 = SII*ANGLE**2+SJJ*(1.-ANGLE**2)+2.*ANGLE*SQRT(1.-ANGLE**2)*
             1SIJ
              IF (AXIS1.LE.0.0) GO TO 211
              AXIS1 = 1./SQRT(AXIS1)
              AXIS2 = (1. - ANGLE**2)*SII+ANGLE**2*SJJ
             1-2.*ANGLE*SQRT(1.-ANGLE**2)*SIJ
              IF (AXIS2.LE.0.0) GO TO 211
              AXIS2 = 1./SQRT(AXIS2)
              SIG = SIGN(1.,(AXIS2-AXIS1)*SIJ)
              LP = UNPLOT(GTEXT)
              CALL RESET(GTEXT,GDSX,GDSY,GPOINT,GDSE,GDSER,GINPUT)
              LL = NV + 1
              NVV = NV*2
              WRITE (4,1660) NAXIS,KK,(R(L),L=LL,NVV)
         1660 FORMAT('P1',2X,'VARIABLE',I2,'AGAINST VECTOR BELOW,(SIMPLE PLANE N
             10.',I2,'SUPERIMPOSED)'/'P    ',10X,8F6.3)
              CALL WRPPTS(GTEXT)
              LP = PLOT(GTEXT)
              C = 3.141593/180.
              THETA = 5.0
              DO 31 K = 1,NV
              DO 31 KK = 1,NG
              ARRKNG ((K-1)*NG+KK) = CLMEAN (KK,K)
           31 CONTINUE
              CALL GMPRD (ARRKNG,R,RARKNG,NG,NV,2)
              DO 100 K=1,NG
              DO 100 KK=1,72
              RADIAN = THETA*C*KK
              ELX = AXIS1*COS(RADIAN)
              ELY = AXIS2*SIN(RADIAN)
              RELPSX(K,KK)=RARKNG(K)+ELX*ANGLE-ELY*SQRT(1.-ANGLE**2)*SIG
```

NOT REPRODUCIBLE

```
FORTRAN IV G LEVEL  19                   RELPSE              DATE = 71257          14/33/21

                 RELPSY(K,KK)=RWRKNG(NG+K)+ELX*SQRT(1.-ANGLE**2)*SIG+ELY*ANGLE
             160 CONTINUE
                 DO 170 K=1,NG
                 DO 170 KK=1,72
                 CALL POINT(GDSER,RELPSX(K,KK),RELPSY(K,KK))
             170 CONTINUE
                 LP = PLCT(GDSER)
                 RETURN
             211 KK = -15
                 RETURN
                 END
```